not only to publish a new translation but to place before his readers a real facsimile of this notable relic of ancient Egyptian mathematics. Through the courtesy of the Trustees of the British Museum photographs of the whole papyrus were made for this work. These are reproduced in twenty-eight plates in volume II, the scale being very little less than the original size. This volume is to contain 111 other plates in two colors, the red and black of the original. Underneath the copied hieratic original is the hieroglyphic transcription, under which is the (right-to-left) transliteration. On the opposite page is a left-to-right arrangement of the transliteration, under which is the literal translation. In this way any one, even though unfamiliar with the Egyptian language or with the scripts used by copyists, may obtain some idea of the meaning of the symbols used and of the language itself.

The first volume contains a free translation of the papyrus with copious notes, an extensive introduction, and an elaborate and critical bibliography, 1706–1926, occupying with indexes more than eighty printed pages. Professor Archibald is alone responsible for the bibliography.

Chancellor Chace is meeting all the expenses of publication, and is presenting the entire edition, with the exception of certain personal gift copies, to the Association. The money received through its sale is to constitute an endowment fund of the Association, and is to be known as the ARNOLD BUFFUM CHACE FUND. The Trustees of the Association desired if possible to set a price for the sale of the work so that the five hundred copies available for this purpose might be rapidly distributed. *A member of the Association may procure the work for personal use* at the special price of fifteen dollars per set, post paid, on ordering it from the Secretary and making a declaration regarding its purchase for personal use. An institutional member may also purchase the work for its library at fifteen dollars per set by ordering it through the Secretary. Other institutions and the public in general may procure it only through the Open Court Publishing Company of Chicago at the regular price of twenty dollars per set. Further announcement will be made and subscription blanks supplied shortly before the date of publication.

An examination of the work will show that the regular price of twenty dollars is much below the actual cost of publication. Members will greatly assist the Association by suggesting that their college and university libraries subscribe to the work and thus aid in the establishment of a fund which will materially assist the Association in its activities.

AFFILIATION OF THE ASSOCIATION OF TEACHERS OF MATHEMATICS IN NEW ENGLAND WITH THE MATHEMATICAL ASSOCIATION OF AMERICA

The council of the Association of Teachers of Mathematics in New England have voted to accept a plan of affiliation originating in 1925 in conferences between President J. L. Coolidge and the officers of that Association. As modified by the Trustees of the Mathematical Association and put into effect by this recent action, it is agreed

1. That the members of the Association of Teachers of Mathematics in New England may become members of the Mathematical Association of America without the payment of the customary initiation fee, the A.T.M.N.E. to supply annually to the Mathematical Association a list of new members whom the Association may invite to their membership under this special condition.

2. That it is understood that whenever the M.A.A. holds meetings in New England, the A.T.M.N.E. engages to aid in every way, and that meetings of the A.T.M.N.E. in connection with those of the M.A.A. as an affiliated organization will be welcomed.

3. That the present agreement may be terminated by either party on six months' notice.

4. That the vote by this council here recorded makes this agreement effective.

W. D. CAIRNS, *Secretary-Treasurer.*

ANNUAL MEETING OF THE ROCKY MOUNTAIN SECTION

The tenth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the Colorado Agricultural College at Fort Collins, Colorado, on April 16, 17. There were thirty-six present including the following seventeen members of the association: A. G. Clark, I. M. DeLong, G. W. Finley, Philip Fitch, J. C. Fitterer, H. C. Gossard, A. J. Kempner, Claribel Kendall, G. H. Light, W. V. Lovitt, S. L. Macdonald, J. Q. McNatt, L. R. Odell, O. H. Rechard, W. J. Risley, H. E. Russell, and C. H. Sisam.

The section voted to hold the next meeting at Colorado College, Colorado Springs. The secretary was instructed to invite the Association to hold a summer meeting at the University of Colorado sometime in the near future.

A committee was appointed consisting of the secretary, Miss Odell, Professor Russell, and Professor Risley to compile material on the significance and value of the study of mathematics. This material is to be published by the section.

The following officers were elected: W. V. LOVITT, chairman; W. J. RISLEY, vice-chairman; PHILIP FITCH, secretary and G. H. LIGHT, treasurer.

The section was favored Friday evening with an address "The New Heavens" by Dr. D. W. Morehouse, president of Drake University.

A complimentary dinner was served on Friday to all present, at which time President C. A. Lory of the Colorado Agricultural College delivered an address of welcome to which Professor I. M. DeLong made an appropriate response.

The following sixteen papers were read:

(1) "Report on the experiment with a standardized test in college algebra" by Professor H. C. GOSSARD.

(2) "The interpretation of errors" by Professor R. L. PARSHALL (by invitation).

(3) "Graphic solutions" by Mr. J. Q. MCNATT.

(4) "Mathematical logic" by Professor H. V. CRAIG (by invitation).

(5) "Concerning d'Alembert's principle" by Professor J. C. FITTERER.

(6) "Finite trigonometric series" by Professor A. G. CLARK.

(7) "Mathematics for freshman women" by Mrs. NELLIE LANDBLOM (by invitation).

(8) "On the reliability of the composite score of a battery test" by Mr. PHILIP FITCH.

(9) "Concerning rigorous proofs" by Professor S. L. MACDONALD.

(10) "Note on the limit functions of sequences of functions of certain types" by Professor O. H. RECHARD.

(11) "The use of the discriminant in differential equations" by Professor G. H. LIGHT.

(12) "Index number bias" by Professor W. V. LOVITT.

(13) "A system of vector coordinates" by Professor H. C. GOSSARD.

(14) "On a property of the Hessians of cubic forms" by Professor C. H. SISAM.

(15) "Complex roots of equations" by Professor A. J. KEMPNER.

(16) "Root extraction with the adding machine" by Professor F. H. LOUD.

In the absence of the author, the abstract of Professor Loud's paper was read by the secretary.

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles.

1. A report of the experiment by the colleges and universities of the Rocky Mountain Section with a standard college algebra speed-accuracy test. This experiment under the direction of the University of Wyoming was voted to be continued through the coming year.

2. This paper dealt with the interpretation of the errors arising in results obtained from experiments on the flow of water over weirs.

3. Mr. McNatt gave some examples of the use of graphic solutions of problems in surveying.

4. The paper discussed the primitive ideas and primitive propositions of the system of Whitehead and Russell; analogies among the calculus of propositions, calculus of classes, and calculus of relations; and the definition of the cardinal numbers 1 and 2. It was shown that a class cannot be used as an argument of anyone of its determining functions and that a hierarchy of functions is necessary.

5. A résumé of its statement and presentation in various texts on mechanics and kinetics was given and it was pointed out that its function, particularly for the teacher of dynamical subjects—especially the practical applications in engineering and allied sciences, consists primarily in simplifying and reducing forces involved in accelerated systems to the field of statics. This service is comparable with the idea Monge had in descriptive geometry, of reducing solid problems to the realm of the plane; and also, as another example, with the conception involved in influence lines in the theory of structures whereby the effect of live loads is simplified to the status of dead loads.

6. Professor Clark indicated that the summation of

$$\sum_{i=1}^{n} F(i) \begin{matrix} \sin \\ \cos \end{matrix} \phi(i)\theta$$

may be effected when $F(i)$ is a rational, integral polynomial by successive application of the parts formula for finite integration,

$$\Sigma u_x \Delta v_x = u_x v_x - \Sigma v_{x+1} \Delta u_x + C_1$$

provided the summation of

$$\sum_{i=1}^{n} \begin{matrix} \sin \\ \cos \end{matrix} \phi(i)\theta$$

is possible.

When $F(i)$ is the quotient of two rational integral polynomials, the summation may be effected by setting up and solving a system of two linear differential equations, the order of the system being the same as the degree of the polynomial forming the denominator.

7. The question as presented by Mrs. Landblom was "What should be the content of a course in mathematics for freshman women and why?" Arguments set forth dealt primarily with problems arising in home life, teaching, special subjects required by curricula, as physics and chemistry, extension work, child welfare, and social settlement. The solution was given in an outline of topics to be covered in a fifty hour course.

8. In this paper it was shown that the composite score of a battery test could be reliable only if the respective scores of its elements were weighted

according to their reliability, validity and independence after having been reduced to equal spread.

9. In his paper Professor Macdonald pointed out the necessity of making conclusions definite in rigorous proofs.

10. An example was given to show that quasi-uniform convergence is not necessary in order that the existence of a limit of a sequence of functions which are pointwise discontinuous shall itself be proved. The following theorem was then proved: A sufficient condition that the limit of a sequence of functions with upper continuity shall have upper continuity is that the sequence is quasi-uniformly convergent.

11. This paper was intended to show how the discriminant can be applied to the solution of differential equations of the type $f(x,y,p)=0$ and showed how the extraneous factor, if any exists, can be detected before solving the equation.

12. Four primary systems of weighting have been devised. In the customary notation these are p_0q_0, p_0q_1, p_1q_0, p_1q_1. The speaker noted the absence of strict mathematical proofs of the assertions made as to the upward or downward bias of an index number weighted with the weights specified above. In this paper some proofs were given of bias when such exists. Professor Irving Fisher has given a proof that the unweighted arithmetic average of relative prices has an upward bias. This paper gave a new proof of this fact.

13. Professor Gossard presented a system of coordinates based upon naming the points (x) of the Gaussian plane by rotations (t) on a base circle plus such translations as called for by the given equation $x=f(t)$. The expressions arising are symmetric functions of (t) and for many types of theorems in geometry the analytic work is exceedingly simple.

14. This paper dealt with the determination of a simplified form for the equation of the Hessian of a cubic in any number of variables.

15. The roots (real and complex) of an equation $w=a_0z^n+a_1z^{n-1}+\cdots a_n\neq 0$, $w=u+iv$, $z=x+iy=re^{i\varphi}$, a_i real or complex, may be isolated and determined to any desired degree of accuracy in the following manner: In a rectangular system of coordinates with a φ-axis and an axis which serves at the same time for $u-$axis and $v-$axis plot, for appropriately chosen values r_i of r, the two curves $u=u(r_i,\varphi)$, $v=v(r_i,\varphi)$. From the order in which the points of intersection of the u-curve with the φ-axis and the points of intersection of the v-curve with the φ-axis follow each other, the number of roots of $w=0$ of absolute value $<r_i$ is read off by a very simple rule. The absolute values of the roots are then determined to any desired degree of accuracy.

Professor Kempner next showed how limits for the arguments of the roots may be determined. The value of the method lies in the fact that instruments

are in existence (harmonic analyzers) which will trace mechanically curves of type $u=u(r_i,\varphi)=\sum c_k \cos k(\varphi)$, $v=v(r_i,\varphi)=\sum d_k \sin k(\varphi)$.

16. This paper dealt with the employment of Newton's binomial formula as a method, and of the adding machine as an instrument, in the extraction of roots. Several cube roots were extracted, as illustrations of certain elementary and rather obvious devices for securing rapid convergence in the series, and otherwise minimizing the labor of computation. In the latter part of the paper, some of the actual records taken from the adding machine were inserted, as fuller demonstration of the method employed.

PHILIP FITCH, *Secretary.*

ELEVENTH ANNUAL MEETING OF THE OHIO SECTION

The eleventh annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, April 2, 1926, in connection with the meetings of the Ohio College Association.

Thirty-four persons registered attendance, among whom were the following twenty-five members of the Association: R. B. Allen, C. L. Arnold, Grace M. Bareis, I. A. Barnett, H. Blumberg, R. D. Bohannan, W. D. Cairns, V. B. Caris, R. Crane, W. Dancer, O. L. Dustheimer, T. M. Focke, B. C. Glover, H. Hancock, H. W. Kuhn, S. E. Rasor, P. L. Rea, Hortense Rickard, S. A. Singer, C. E. Stout, J. H. Weaver, R. B. Wildermuth, F. B. Wiley, J. B. Winslow, B. F. Yanney.

Officers elected for the coming year were: Chairman, H. W. KUHN; Secretary-Treasurer, RUFUS CRANE; Member of Executive Committee, H. M. BEATTY; Member of Program Committee, T. M. FOCKE. A resolution was adopted expressing the appreciation of the members of the section for the life and services of the late Professor G. N. Armstrong, and their regret at his death. A committee was appointed to study the advisability of attempting to organize a new section of the American Mathematical Society, centering in Ohio. A committee was appointed to study ways and means of improving the teaching situation in the secondary schools. It is expected that the next meeting will be held on April 8, 1927.

The following papers were read:

(1) "Alphabetic symbolism applied to some operations on power series" by the Chairman, Professor R. D. BOHANNAN, Ohio State University.

(2) "Euclidean invariants of plane second degree curves" by Professor C. C. MACDUFFEE, Ohio State University.

(3) "Controversial mathematics" by Professor H. BLUMBERG, Ohio State University.

(4) "A major in mathematics" by Professor O. L. DUSTHEIMER, Baldwin-Wallace College.

(5) "The advisability of organizing a section of the American Mathematical Society in this part of the country" by Professor HARRIS HANCOCK, University of Cincinnati.

(6) "Qualifications of teachers in the subjects which they teach" by Professor HANCOCK.

In the absence of Professor MacDuffee, the second paper was read by Mr. L. J. Paradiso, student at Ohio State University. Abstracts of some of these papers follow:

2. This paper has already appeared in full in the MONTHLY (*1926*, 243–252).

3. Professor Blumberg discussed the difficulties that have arisen, especially since the appearance of Cantor's work on the transfinite, in connection with the notions of "set," "infinity," and "existence," and sketched the theories of Zermelo, Brouwer, Weyl, and Hilbert, that were projected to meet such difficulties.

4. Professor Dustheimer presented a summary of replies from 136 institutions to a questionnaire, which showed that about 50 per cent of the colleges require 2 units of high school mathematics for entrance, while about 30 per cent require $2\frac{1}{2}$ units; 50 per cent require mathematics for graduation; only 34 per cent require solid geometry of students majoring in mathematics; 53 per cent give college credit for solid geometry; 25 per cent give college credit for intermediate algebra; an average of 7 per cent of seniors specialize in mathematics; this percentage is increasing in 55 per cent of the colleges. According to this survey, the following would be a good major in mathematics: college algebra 3 hours, trigonometry 3 hours, analytic geometry 3 hours, calculus 6 hours, differential equations 3 hours, history of mathematics 2 hours, theory of equations or some applied mathematics 3 hours, and teaching of mathematics 2 hours.

5. Professor Hancock paid high tribute to what has been accomplished by the American Mathematical Society, stating that thereby the mathematical standard of excellence in every university of the country has been greatly elevated. Professor Hancock said that there are at the present time more good mathematicians within the area that would be included by the section in question than there were thirty years ago in the whole country. He declared that such a section would give a greater impetus to mathematical research and would thereby create a greater interest in mathematics in the Ohio Valley Section, and that such increased interest would eventually strengthen the society as a whole.

6. Professor Hancock called attention to the exceedingly low marks that were made by freshmen in mathematics in examinations that were held last year in three of the larger universities of the state. He ascribed this in great measure

to the modern conditions that have existed for the last twenty years. To remedy such conditions he proposed:

That teachers of mathematics in the secondary schools who have majored in mathematics, and who hold an A.B. or some similar degree, be put on a more advanced salary scale than those teachers of mathematics who are not so qualified.

That those teachers of mathematics who have attained the A.M. degree in mathematics be put on a still higher scale, and that the highest salary scale should be reserved for those teachers who have majored in mathematics with the degree of Doctor of Philosophy.

That those teachers who have no such degrees be put on similar scales of salary and promotions, if they have already shown themselves able teachers and if they pass examinations in mathematics equivalent to the examinations leading to the A.M. and Ph.D. degrees.

RUFUS CRANE, *Secretary-Treasurer.*

THIRD ANNUAL MEETING OF THE MICHIGAN SECTION

The third annual meeting of the Michigan section of the Mathematical Association of America was held at the University of Michigan, Ann Arbor, on April 1, 1926. Chairman E. R. Sleight called the meeting to order at 9 A.M.

The attendance was fifty-six including the following thirty members of the Association: N. H. Anning, P. N. Blessing, J. W. Bradshaw, H. C. Carver, R. W. Clack, C. C. Craig, W. W. Denton, B. F. Dostal, P. Field, S. E. Field, W. B. Ford, J. W. Glover, L. A. Hopkins, M. F. Johnson, L. C. Karpinski, D. K. Kazarinoff, H. B. Lemon, T. Lindquist, A. L. Nelson, H. L. Olson, J. R. Overman, O. J. Peterson, V. C. Poor, T. E. Raiford, C. Reid, T. R. Running, J. Shohat, E. W. Sleight, C. C. Spooner.

At a short business session, reports of the retiring officers were received and the following new officers were elected: Chairman, A. L. NELSON, College of the City of Detroit; Secretary-Treasurer, C. REID, University of Michigan; Member of Executive Committee, H. L. OLSON, Michigan State College.

The following papers were presented, those numbered 6 and 8 being read by title. Correspondingly numbered abstracts of the others follow.

(1) "Some elementary applications of the theory of correlation" by Mr. CLAIR REID, University of Michigan.

(2) "Stepping up to get curl and divergence" by Professor W. W. DENTON, University of Michigan.

(3) "On Legendre polynomials" by Professor J. SHOHAT, University of Michigan.

(4) "Conditions for a fixed point in projective differential geometry" by Professor A. L. NELSON, College of the City of Detroit.

(5) "On Pappus's problem" by Mr. D. K. KAZARINOFF, University of Michigan.

(6) "A focal point construction for a certain type of problem in the calculus of variations" by Professor V. G. GROVE, Michigan State College.

(7) "Parallel surfaces with straight line orthogonal trajectories and orthogonal curvilinear coordinates" by Professor W. S. KIMBALL, Michigan State College (by invitation).

(8) "A consideration of the relativity of magnitudes" by Mr. C. E. SMITH, Northwestern High School, Detroit (by invitation).

1. In this paper, Mr. Reid gave four applications of the theory of correlation. The first two were examples of the correct use of correlation coefficients. The third was a case in which there was perfect correlation although the linear coefficient was zero. The fourth illustration showed a high linear correlation between two sets of variates not directly correlated but correlated with a hidden third variate.

2. In elementary textbooks on mathematical physics, the proofs of Stokes' formula, Gauss' formula, the relation of curl to angular velocity, the equation of continuity, and similar theorems are, in general, rather poorly presented. The proofs in question involve the expansion of functions by Taylor's formula and integration of results over an infinitesimal curve or surface. The object of Professor Denton's paper was to explain why these particular proofs are weak and to point out how they may be strengthened. The improvements suggested depend upon the following device. With each point of the given vector field there is associated an auxiliary system of coordinates having that point as origin, and the integration over the infinitesimal curve or surface is actually carried out.

3. Professor Shohat showed that the asymptotic expression (for $n \to \infty$) of the Legendre polynomial $X_n(x)$ can be derived directly from the difference equation

$$(n+2)X_{n+2}(x) - (2n+3)xX_{n+1}(x) + (n+1)X_n(x) = 0,$$

by a method similar to that given by Liouville (*Journal des Mathématiques*, vol. II (1837), pp. 16–35), and by W. B. Ford (*Transactions, American Mathematical Society*, vol. XV (1909), pp. 319–336). An important point is the proof of the formula

$$\frac{2 \cdot 4 \cdot \cdots \quad 2n}{1 \cdot 3 \cdot \cdots (2n-1)} = \sqrt{\pi\left(n + \tfrac{1}{4} + O(n^{-1})\right)}.$$

4. In two special theories of Wilczynski's projective differential geometry, it has been noted that one of the coordinates of a fixed point with reference to a local tetrahedron must be a solution of differential equations geometrically adjoint to the basic system. Professor Nelson explained and generalized this fact. The paper will appear in the *Bulletin of the American Mathematical Society*.

5. To give a direct proof of the theorem: "If a triangle have two equal angle-bisectors then it will have two equal sides" is a problem which has recently attracted the interest of readers of the MONTHLY (*1917*, 344; *1918*, 182). The history of the problem was traced and it was pointed out that a neat solution follows from use of a construction given by Pappus for placing between the arms of a given angle a line-segment which shall be bisected at a given point.

7. As a result of an investigation of the potential theory of thermionic currents where the electrons pursue straight line paths, Professor Kimball was led to the three following theorems:

I. A necessary and sufficient condition for $F(x,y,z)=a$ to be a family of surfaces having straight lines for its orthogonal trajectories is that h (Lamé's first differential parameter), be a function of a only: $h=\sqrt{F_x^2+F_y^2+F_z^2}=f(a)$.

II. These surfaces are identified with families of parallel surfaces.

III. These surfaces are shown to be "Families of Lamé": they may be one of three orthogonal curvilinear coordinates, the other two of which turn out to be families of developable surfaces.

NORMAN ANNING, *Secretary-Treasurer*.

ON THE ASYMPTOTIC EXPRESSIONS FOR JACOBI AND LEGENDRE POLYNOMIALS DERIVED FROM FINITE-DIFFERENCE EQUATIONS[1]

By J. A. SHOHAT, University of Michigan

1. Introduction. We denote by

$$\varphi_n(x)\equiv\varphi_n(x\,;\ \alpha,\beta)=a_n(\alpha,\beta)x^n+\cdots\ (n=0,1,2\cdots\ ;\ a_n>0) \tag{1}$$

$$\left[\varphi_n'\equiv\varphi_n'(x)\equiv\frac{\varphi_n(x;\alpha,\beta)}{a_n(\alpha,\beta)}=x^n+\cdots\right]$$

---

[1] Presented to the American Mathematical Society, Kansas City, December, 1925, and (on Legendre polynomials) to the Michigan Section of the Mathematical Association of America, Ann Arbor, April, 1926.

the system of orthogonal and normal polynomials of Jacobi corresponding to the interval $(-1, 1)$, with the characteristic function $(1+x)^{\alpha-1}(1-x)^{\beta-1}$, $(\alpha, \beta>0)$. We have then,[1]

$$\int_{-1}^{1}(1+x)^{\alpha-1}(1-x)^{\beta-1}\varphi_n(x)\varphi_m(x)dx=\begin{cases}0, & m\neq n,\\ 1, & m=n;\end{cases} \tag{2}$$

$$a_n(\alpha,\beta)=\sqrt{\frac{(\alpha+\beta+n-1)\cdots(\alpha+\beta+2n-2)\Gamma(\alpha+\beta+2n)}{\Gamma(\alpha+n)\Gamma(\beta+n)\Gamma(n+1)}}\cdot 2^{-n-\frac{\alpha+\beta-1}{2}}$$

$$=2^{n+\frac{\alpha+\beta-1}{2}}\sqrt{\frac{1}{\pi}}\,(1+o(1)). \tag{3}$$

[We use here the formula:

$$p(p+1)\cdots(p+n)=\frac{n^p\Gamma(n+1)(1+o(1))}{\Gamma(p)},\ p>0,\ n>0-\text{integer.}\Big]$$

$$\varphi_n'(1\ ;\alpha,\beta)=\frac{2^n\beta(\beta+1)\cdots(\beta+n-1)}{(\alpha+\beta+n-1)\cdots(\alpha+\beta+2n-2)},$$

$$\varphi_n'(-1\ ;\alpha,\beta)=\frac{(-1)^n2^n\alpha(\alpha+1)\cdots(\alpha+n-1)}{(\alpha+\beta+n-1)\cdots(\alpha+\beta+2n-2)}. \tag{4}$$

Special case:

$$\alpha=\beta=1\ ;\ \varphi_n(x\ ;1,1)=\sqrt{\frac{2n+1}{2}}\ X_n(x)\text{—Legendre polynomial.} \tag{5}$$

For any function $f(x)$ we have the formal development,

$$f(x)\sim\sum_0^\infty A_i\varphi_i(x),\quad A_i=\int_{-1}^{1}(1+x)^{\alpha-1}(1-x)^{\beta-1}f(x)\varphi_i(x)dx, \tag{6}$$

provided, of course, the integrals A_i exist. The convergence of the development (6) depends essentially upon the behavior of $\varphi_n(x)$ for n very large. This has been investigated by Darboux,[2] Stekloff,[3] W. B. Ford[4] and others.

---

[1] W. Stekloff, Sur l'approximation des fonctions, *Bulletin of the Russian Academy of Sciences* (1917), 187–218; p. 211.

[2] Darboux, Mémoire sur l'approximation des fonctions de très grandes nombres, *Journal des Mathématiques*, t. IV, sér. III (1878), 5–57, 377–416; pp. 23–24.

[3] W. Stekloff, Sur les expressions asymptotiques, *Transactions of the Kharkoff Mathematical Society* (1907), 1–103; pp. 21–27.

[4] W. B. Ford, On the integration of the homogeneous linear difference equation of second order, *Transactions of the American Mathematical Society*, 2nd ser., v. XV (1909), 319–336; pp. 335–6.

In this paper *we derive the asymptotic expression (for $n\to\infty$) of $\varphi_n(x;\alpha,\beta)$ in an elementary way, directly from the fundamental difference-equation satisfied by $\varphi_n(x;\alpha,\beta)$.* The method used is of the same character as that developed by Liouville[1] and Ford (*loc. cit.*).

2. Jacobi polynomials in general. We get readily from (2) a difference-equation for $\varphi'_n(x)$,

$$\varphi'_{n+2}(x) - (x-c_{n+2})\varphi'_{n+1}(x) + \lambda_{n+2}\varphi'_n(x) = 0 \quad (n \geqq 0), \tag{7}$$

where c_{n+2}, λ_{n+2} are constants, which do not depend on x. Making use of (4), we get

$$\begin{aligned} c_{n+2} &= 1 - \tfrac{1}{2}\frac{(n+\alpha+\beta)\left(n+\frac{\alpha+\beta}{2}\right)(n+\beta+1) + \left(n+\frac{\alpha+\beta}{2}+1\right)(n+\alpha)(n+1)}{\left(n+\frac{\alpha+\beta}{2}\right)\left(n+\frac{\alpha+\beta}{2}+\tfrac{1}{2}\right)\left(n+\frac{\alpha+\beta}{2}+1\right)} \\ &= \frac{(\alpha-\beta)(\alpha+\beta-2)}{4n^2} + \cdots, \\ \lambda_{n+2} &= \frac{1}{4}\frac{(n+1)(n+\alpha)(n+\beta)(n+\alpha+\beta-1)}{\left(n+\frac{\alpha+\beta}{2}\right)^2\left(n+\frac{\alpha+\beta}{2}-\tfrac{1}{2}\right)\left(n+\frac{\alpha+\beta}{2}+\tfrac{1}{2}\right)} \\ &= \frac{1}{4}\left(1 + \frac{-2\alpha^2-2\beta^2+4\alpha+4\beta-3}{4n^2} + \cdots\right), \end{aligned} \tag{8}$$

$$c_{n+2} = \frac{-\delta'_n}{n^2}, \quad \lambda_{n+2} = \tfrac{1}{4} - \frac{\delta_n''}{n^2}, \quad \delta'_n, \delta''_n \text{ finite for } n\to\infty\ ;$$

$$\varphi'_{n+2} - x\varphi'_{n+1} + \tfrac{1}{4}\varphi'_n = \frac{1}{n^2}\left[\delta'_n\varphi'_{n+1} + \delta''_n\varphi'_n\right]. \tag{9}$$

Consider the interval

$$-1+\epsilon \leqq x \leqq 1-\epsilon \quad (\epsilon > 0 \text{ arbitrarily small, but fixed}). \tag{10}$$

We get in (10) as solution of the homogeneous equation

$$z_{n+2} - xz_{n+1} + \tfrac{1}{4}z_n = 0,$$

$$z_n = \frac{1}{2^{n-1}}\left\{C'\cos n\varphi + C''\frac{\sin n\varphi}{\sin\varphi}\right\} \quad (x = \cos\varphi\ ;\ 0 < \varphi < \pi). \tag{11}$$

---

[1] Liouville, Second mémoire sur le developpement des fonctions, *Journal des Mathématiques*, t. II (1837), pp. 23–4.

Therefore, the solution of the non-homogeneous equation (9) is

$$\varphi_n' = \frac{1}{2^{n-1}}\left\{ C_n' \cos n\varphi + C_n'' \frac{\sin n\varphi}{\sin \varphi} \right\}, \tag{12}$$

where C_n', C_n'' are functions of n such that

$$\Delta C_n' \cos (n+1)\varphi + \Delta C_n'' \frac{\sin (n+1)\varphi}{\sin \varphi} = 0, \quad (\Delta C_n = C_{n+1} - C_n),$$

$$\Delta C_n' \cos (n+2)\varphi + \Delta C_n'' \frac{\sin (n+2)\varphi}{\sin \varphi} = \frac{2^{n+1}}{n^2} (\delta_n' \varphi_{n+1}' + \delta_n'' \varphi_n'), \tag{13}$$

$$C_n' = C' - \sum_1^n \frac{2^{n+1}}{n^2} (\delta_n' \varphi_{n+1}' + \delta_n'' \varphi_n') \frac{\sin (n+1)\varphi}{\sin \varphi},$$

$$C_n'' = C'' + \sum_1^n \frac{2^{n+1}}{n^2} (\delta_n' \varphi_{n+1}' + \delta_n'' \varphi_n') \cos (n+1)\varphi, \tag{14}$$

C', C'' do not depend on n.

Taking in (6) $f(x) = \varphi_n(x; \alpha, \beta)$, $\varphi_i(x) = \varphi_i(x; 1, 1)$ and using the known property: $|X_n(x)| \leqq 1$ for $-1 \leqq x \leqq 1$, Stekloff shows[1] that

$$2^{n-1} \varphi_n'(x\,;\, \alpha, \beta) = O(n^{1/2}), \qquad (-1+\epsilon \leqq x \leqq 1-\epsilon). \tag{15}$$

(15) assures the uniform and absolute convergence of

$$\sum_1^\infty \frac{2^{n+1}}{n^2} (\delta_n' \varphi_{n+1}' + \delta_n'' \varphi_n') \begin{matrix} \cos \\ \sin \end{matrix} (n+1)\varphi,$$

and we get from (12, 14),

$$C_n' = C_1 + \frac{1}{\sin\varphi} \sum_n^\infty (a_n' \varphi_{n+1}' + a_n'' \varphi_n'),$$

$$C_n'' = C_2 + \sum_n^\infty (b_n' \varphi_{n+1}' + b_n'' \varphi_n'); \tag{16}$$

C_1, C_2 do not depend on n; a_n', a_n'', b_n', b_n'' are finite for $n \to \infty$.

$$\varphi_n'(\cos\varphi) = \frac{C_1 \cos n\varphi + C_2 \sin n\varphi / \sin \varphi}{2^{n-1}}$$

$$+ \frac{\cos n\varphi}{2^{n-1} \sin \varphi} \sum_n^\infty \frac{2^{n+1}}{n^2} (a_n' \varphi_{n+1}' + a_n'' \varphi_n')$$

$$+ \frac{\sin n\varphi}{2^{n-1}} \sum_n^\infty (b_n' \varphi_{n+1}' + b_n'' \varphi_n') \frac{2^{n+1}}{n^2}. \tag{17}$$

[1] *Loc. cit.* (2), p. 205.

The two sums $\sum_n^\infty$ in (17) become infinitely small with $1/n$. Therefore, (17) shows, in the first place, that $2^{n-1}\varphi_n(x)$ *is finite for* $n\to\infty$ *in the interval* (10). But we can go further. We notice in general that

$$U_n = O\left(\frac{1}{n^{3/2+\alpha}}\right) \text{ with } \alpha \geqq 0 \text{ implies : } \sum_n^\infty U_n = O\left(\frac{1}{n^{1/2+\alpha}}\right). \tag{18}$$

This, applied to (17), gives (since in the sums $\sum_n^\infty$ under consideration the first terms are of order $1/n^2$) $1/n$ *as the order of each sum.* We get finally, using (3),

$$\left.\begin{aligned} \varphi_n'(x\,;\alpha,\beta) &= \frac{A_1\cos n\varphi + A_2\sin n\varphi + O\left(\frac{1}{n}\right)}{2^{n-1}} \\ \varphi_n(x\,;\alpha,\beta) &= B_1\cos n\varphi + B_2\sin n\varphi + O\left(\frac{1}{n}\right) \end{aligned}\right| \begin{matrix} x=\cos\varphi, \\ (-1+\epsilon\leqq x\leqq 1-\epsilon), \end{matrix} \tag{19}$$

A_1, A_2, B_1, B_2 do not depend on n.

This is the required asymptotic expression for Jacobi polynomials with arbitrary positive parameters α, β. We may add, that (4) gives

$$\varphi_n(1\,;\alpha,\beta) = \frac{2^{1-\frac{\alpha+\beta}{2}}n^{\beta-\frac{1}{2}}(1+o(1))}{\Gamma(\beta)}\,;\ \varphi_n(-1\,;\alpha,\beta)$$
$$= (-1)^n\frac{2^{1-\frac{\alpha+\beta}{2}}n^{\alpha-\frac{1}{2}}(1+o(1))}{\Gamma(\alpha)}.$$

Note: (8) shows that $c_n\equiv 0$ for $\alpha=\beta$, and that $\lambda_n=\frac{1}{4}$, $c_n=0$ simultaneously, if and only if

$$-2\alpha^2-2\beta^2+4\alpha+4\beta-3=0,\quad (\alpha-\beta)(\alpha+\beta-2)=0$$

i.e., $O(1/n)\equiv 0$ in (19) in the following cases[1] only:

$$\alpha=\beta=\tfrac{1}{2}\,;\quad \varphi_n(\cos\varphi)=\sqrt{\frac{2}{\pi}}\cos n\varphi\left(n\geqq 1\,;\ \varphi_0=\sqrt{\frac{1}{\pi}}\right)$$
$$\alpha=\beta=\frac{3}{2}\,;\quad \varphi_n(\cos\varphi)=\sqrt{\frac{2}{\pi}}\frac{\sin(n+1)\varphi}{\sin\varphi} \tag{20}$$
$$\alpha=\frac{3}{2},\beta=\tfrac{1}{2}\,;\varphi_n(\cos\varphi)=\sqrt{\frac{1}{\pi}}\frac{\cos(2n+1)\varphi/2}{\cos\varphi/2}$$
$$\alpha=\tfrac{1}{2},\beta=\frac{3}{2}\,;\varphi_n(\cos\varphi)=\sqrt{\frac{1}{\pi}}\frac{\sin(2n+1)\varphi/2}{\sin\varphi/2}.$$

[1] To obtain (20), we use (2), substituting $x=\cos\phi$.

3. Polynomials of Legendre. (19) holds for Legendre polynomials, as a particular case ($\alpha=\beta=1$; see (5)). It seems to be of interest, however, to treat Legendre polynomials directly from the classical difference equation

$$X_{n+2}(x) - \frac{2n+3}{n+2} X_{n+1}(x) + \frac{n+1}{n+2} X_n(x) = 0, \ (n \geqq 0). \tag{21}$$

The method developed above obviously cannot be applied to (21). We apply, therefore, to (21) a certain transformation, which may be useful in similar problems. We introduce, instead of $X_n(x)$, a new function

$$U_n(x) \equiv \frac{X_n(x)}{\theta_n}, \tag{22}$$

where θ_n depends on n only and satisfies the relation

$$\theta_{n+2} = \frac{n+1}{n+2}\theta_n, \tag{23}$$

so that the transformed equation (21) for $U_n(x)$ is

$$U_{n+2} - \frac{2n+3}{n+1}\frac{\theta_{n+1}}{\theta_n} xU_{n+1} + U_n = 0. \tag{24}$$

We must find the asymptotic expression of θ_n for n very large. (23) gives, with two arbitrary constants θ_0, θ_1,

$$\theta_{2m} = \frac{1\cdot 3\cdots(2m-1)}{2\cdot 4\cdots 2m}\theta_0; \ \theta_{2m+1} = \frac{2\cdot 4\cdots 2m}{1\cdot 3\cdots(2m+1)}\theta_1. \tag{25}$$

We write now, with Tchebycheff, Wallis' formula as follows:

$$\frac{\pi}{2} = \left[\frac{2\cdot 4\cdots 2m}{1\cdot 3\cdots(2m-1)}\right]^2 \cdot \frac{Y}{2m+1} = \left[\frac{2\cdot 4\cdots 2m}{1\cdot 3\cdots(2m-1)}\right]^2 \cdot \frac{X}{2m}, \tag{26}$$

$$Y = \frac{2m+2}{2m+1}\cdot\frac{2m+2}{2m+3}\cdots = \left(1+\frac{1}{(2m+1)(2m+3)}\right)\left(1+\frac{1}{(2m+3)(2m+5)}\right)\cdots,$$

$$X = \frac{2m}{2m+1}\cdot\frac{2m+2}{2m+1}\cdots = \left(1-\frac{1}{(2m+1)^2}\right)\left(1-\frac{1}{(2m+3)^2}\right)\cdots. \tag{27}$$

Tchebycheff[1] gets, using the fact, that $Y>1$, $X<1$,

$$\left[\frac{2\cdot 4\cdots 2m}{1\cdot 3\cdots(2m-1)}\right]^2 = \frac{\pi(m+\frac{1}{2})}{Y} = \frac{\pi m}{X} = \pi(m+\epsilon_m), \ (0<\epsilon_m<\tfrac{1}{2}). \tag{28}$$

[1] Tchebycheff, On functions deviating the least from zero, *Collected Papers* (in Russian) t. II, 189–215; p. 209.

The asymptotic expression (28) *is insufficient for our analysis.* We must estimate ϵ_m. We proceed to prove:

$$\lim_{m\to\infty} \epsilon_m = \tfrac{1}{4}\ ; \frac{2\cdot 4\cdots 2m}{1\cdot 3\cdots(2m-1)} = \sqrt{\pi\left(m+\tfrac{1}{4}+O\left(\frac{1}{m}\right)\right)}. \tag{29}$$

In order to obtain (29), we investigate X or Y in (26, 27). We get from (27), using the inequalities

$$\log(1+x) = \frac{x}{1+\theta x}\,(0<\theta<1),\ x > \log(1+x) > \frac{x}{1+x}\,(x>0),$$

$$\begin{aligned} &\frac{1}{(2m+1)(2m+3)} + \frac{1}{(2m+3)(2m+5)} + \cdots > \log Y \\ &> \frac{1}{1+(2m+1)(2m+3)} + \frac{1}{1+(2m+3)(2m+5)} + \cdots. \end{aligned} \tag{30}$$

We get further, since

$$\frac{1}{(2m+1)(2m+3)} + \frac{1}{(2m+3)(2m+5)} + \cdots = \frac{1}{2(2m+1)} = \frac{1}{4m+2},$$

$$\begin{aligned} &\frac{1}{1+(2m+1)(2m+3)} + \frac{1}{1+(2m+3)(2m+5)} + \cdots \\ &= \tfrac{1}{4}\sum_{1}^{\infty}{}_{i}\ \frac{1}{(m+i)^2} > \tfrac{1}{4}\int_{m+1}^{\infty}\frac{dx}{x^2} = \frac{1}{4m+4}, \end{aligned}$$

$$\log Y = \frac{1}{4(m+\theta)}\ ;\ Y = e^{\frac{1}{4(m+\theta)}}\ (\tfrac{1}{2} < \theta < 1). \tag{31}$$

(31), substituted into (28), gives (29), or

$$\frac{1\cdot 3\cdots(2m-1)}{1\cdot 2\cdots\quad m} = 2^m\sqrt{\frac{1}{\pi\left(m+\tfrac{1}{4}+O\left(\frac{1}{m}\right)\right)}}. \tag{32}$$

(25) gives now

$$\theta_{2m} = \theta_0\sqrt{\frac{1}{\pi\left(m+\tfrac{1}{4}+O\left(\frac{1}{m}\right)\right)}},\quad \theta_{2m+1} = \theta_1\frac{\sqrt{\pi\left(m+\tfrac{1}{4}+O\left(\frac{1}{m}\right)\right)}}{2m+1},$$

and with $\theta_0=\pi$, $\theta_1=2$, we get a single formula

$$\theta_n = \sqrt{\frac{2\pi}{n}}\left(1+O\left(\frac{1}{n}\right)\right). \tag{33}$$

(24), combined with (33), gives:

$$U_{n+2}(x) - 2xU_{n+1}(x) + U_n(x) = \frac{x\delta_n'}{n^2} U_{n+1}(x), \qquad (34)$$

$$\delta_n' \text{ finite for } n \to \infty,$$

an equation quite similar to (9), *which can be treated similarly*, since

$$U_n(x) = \frac{X_n(x)}{\theta_n} = \sqrt{\frac{n}{2\pi}}\, X_n(x)\left(1 + O\left(\frac{1}{n}\right)\right);$$

$$|\, U_n(x)\,| = O(n^{\frac{1}{2}}) \text{ for } -1 \leqq x \leqq 1. \qquad (35)$$

We get then for any x in $(-1+\epsilon,\ 1-\epsilon)$ an asymptotic expression of $U_n(x)$ similar to (19), and then, by means of (35),

$$X_n(\cos\varphi) = \sqrt{\frac{2\pi}{n}}\left(C_1 \cos\varphi + C_2 \sin\varphi + O\left(\frac{1}{n}\right)\right), \qquad (36)$$

where C_1, C_2 do not depend on n, $0 < \epsilon \leqq \varphi \leqq \pi - \epsilon$.

SOME APPLICATIONS OF MATHEMATICS TO ARCHITECTURE: GOTHIC TRACERY CURVES[1]

By E. C. PHILLIPS, S. J., Georgetown University

1. Some fifteen years ago Professor Frank Morley of Johns Hopkins University suggested as an interesting problem the study of rose-window curves. In this paper, which was inspired by that suggestion, an attempt is made to express by means of mathematical equations some of those beautiful forms of architectural adornment which give to the great cathedrals and other buildings of the middle ages much of the graceful detail which has won and held the admiration of succeeding centuries. As an introduction to the subject a brief historical account of the origin and development of Gothic architectural decoration will be given. Much has been written on this subject which has been treated at length in most of the histories of architecture, but a very brief summary is sufficient for our purpose and the following notes have been extracted chiefly from an excellent article on "The Gothic Window" by P. R. McCaffrey, O.C.C., published in *The Ecclesiastical Review* for March, 1924, (vol. LXX, no. 3, pp. 254–269).

2. In the earliest stone buildings of medieval times, when the houses of the great were fortresses as well as dwellings, the windows were small rectangular

[1] Presented at the Kansas City meeting of the Association, December 31, 1925.

apertures which were splayed inward in order to help diffuse the light while the narrowness of the external opening served as a protection against attack from without. Next round windows, either entirely circular or with a semi-circular head on a rectangular body came into use; then two round windows were made adjacent as in Norman and Romanesque architecture, while a later modification placed a circular window above two adjacent ones, as is found in certain Cistercian churches.

A subsequent development of great importance was the *lancet*, a high narrow window with a pointed arch at the top, the two sides of the arch being arcs of circles of relatively long radius, and in this type of construction we find the origin of the later Gothic style which tends ever towards the heavens and invites us to raise our minds and hearts above the sordid things of earth. After the development of the single lancet, larger and more lightsome windows were constructed by bringing lancets together in twos, threes, fives or sevens and enclosing these multiple lancets under a single external arch. After the advent of the double lancet with its enclosing common arch or drip stone, *plate tracery* was introduced which, in its earliest form, consisted of a single circular hole pierced in the solid stone slab between the two lancets and the external arch. Later, this single circle was replaced by trefoils, quatrefoils and other forms which finally became very intricate. This style of construction or decoration is called plate tracery because the opening was cut in a single slab or plate of stone, or at least in a flat solid portion of the wall; but the architect soon rose above this limitation and began to build up his tracery by assembling into the structure of the window many more slender pieces of carved stone and thus originated what is called *bar tracery*; it is this kind of construction which is generally understood when we speak of Gothic tracery. The bars are the mullions, or solid portions, separating the lights, or clear portions, of the window and forming the framework which supports the glass. Gothic architecture dates from the twelfth century, its foundations being laid in 1144 when Abbot Suger completed the great Royal Abbey of St. Denis, near Paris, and mullions were first introduced towards the end of this same century, reaching their highest perfection in the middle of the thirteenth century. The designs of the tracery became more and more intricate and it is truly wonderful what lightness and grace were secured by the artists and artisans of those days who designed and chiseled and fitted together the hundreds of pieces of stone going to make up the framework of the great windows of the thirteenth and fourteenth centuries, as for example that of the marvelous rose windows in the transepts of the Cathedral of Notre Dame in Paris.

The designs of all the earlier Gothic windows consisted entirely of combinations of straight lines and circular arcs or complete circles, but other curves were introduced later and hence we have three main divisions of Gothic

tracery, known respectively as (1) geometrical tracery, (2) curvilinear or flowing tracery and (3) flamboyant tracery. The characteristics of these three styles may be summed up briefly as follows:

(1) In geometrical tracery all the lines are straight or circular, and hence, while the curves are mechanically continuous, they cannot be represented by any single continuous function of the coordinates, each part of the curve requiring a separate equation; there are no true cusps or flexes but we pass from straight line to circular arc or from one circular arc to another.

(2) In curvilinear or flowing tracery the architect made all the portions of the main design to flow on uninterruptedly, the apices formed in geometrical tracery by the juncture of two circular arcs being now replaced by true cusps and the juncture of a straight line and a circular arc or of two oppositely curved circular arcs becoming a true flex.

(3) In the flamboyant style the curves became more spiral in form and took on somewhat the aspect of tongues of flame. This style is considered as a decadent type of Gothic and was used chiefly on the continent while in England the flowing Gothic was being supplanted by the Tudor or rectangular style; there are however some beautiful examples of flamboyant tracery found in England, one of the best being that of the great west window of York Cathedral.

3. The discussion in this paper is restricted to flowing and flamboyant tracery since only they employ continuous curves having continuous first and second derivatives, though some examples of geometrical tracery contain designs which in appearance differ almost imperceptibly from such curves. In their general character most of these curves, in all three styles, resemble a flat flower with pointed petals arranged uniformly about a circular pistil, and our purpose is to find and discuss a system of equations which will represent the varied forms found in these circular designs.

The method of attack which suggested itself was first to find, in rectangular Cartesian coordinates, the equation of a periodic curve something like the cycloid but having cusps both above and below instead of concave arches bounded by cusps. When such a curve was obtained the next step was to bend this curve in its own plane into circular form so that each cuspidal arch should become a petal of our flower or rose-window. Several equations were tried but the simplest and most convenient was the following one given in parametric form:

$$x=(t-\tfrac{1}{2}\sin 2t)/p, \qquad y=1+k\cos t, \tag{1}$$

p being an arbitrary constant, and $k^2 \leqq 1$. On differentiating we find

$$p\,dx/dt=1-\cos 2t=2\sin^2 t, \qquad dy/dt=-k\sin t; \tag{2}$$

$$p\,d^2x/dt^2=2\sin 2t=4\sin t\cos t, \qquad d^2y/dt^2=-k\cos t. \tag{3}$$

Thus dx/dt and dy/dt are both zero whenever $\sin t$ is zero, that is for $t=0$, π, 2π, 3π, ... $n\pi$; whilst the second derivatives are simultaneously zero only when $\cos t$ is zero, that is for $t=\pi/2$, $3\pi/2$, ... $(2n-1)\pi/2$. Hence there is a cusp for $t=0$, π, 2π, ... $n\pi$; and a flex for $t=(2n-1)\pi/2$. The cusps for $t=0$, 2π, 4π, etc., all point upwards and lie on the line $y=1+k$; whilst those for $t=\pi$, 3π, 5π, etc., point downwards and lie on the line $y=1-k$. The flexes are midway between the cusps and lie on the line $y=1$. The form of the curve is shown in Fig. 1.

FIG. 1

If we now consider this strip bounded by the line $y=1\pm k$ to be flexible in the plane of the paper so that we can take any length of it and bend it into circular form, the cuspidal tangents $x=0$, π, 2π, etc., will go into concurrent straight lines, and the system of straight lines $y=$any constant, will go into concentric circles with the transforms of the x-lines as radii; each cuspidal arch of the original curve will then form a cuspidal petal and we have a window tracery of attractive appearance as shown in Fig. 2, which is the result of bending three or more arches into the circular form.

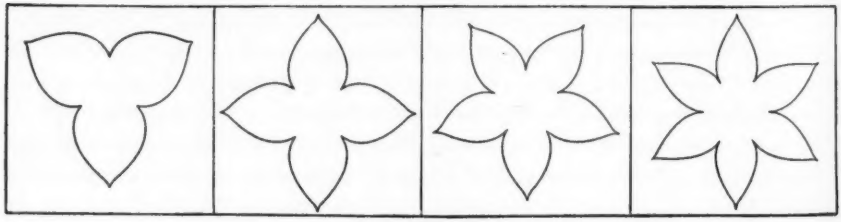


FIG. 2

In order to perform this wrapping process mathematically we merely substitute in equations (1) θ for x and ρ for y, where θ and ρ are polar coordinates; and in these new equations p will be the number of petals in our design. Hence the required equations are:

$$\theta = (1/p)(t - \tfrac{1}{2}\sin 2t), \qquad \rho = 1 + k\cos t. \tag{4}$$

For the present we will restrict the values of p to positive integers, so that we will have p entire petals in the circuit of 360°. Since the form of the differential equations (2) and (3) is unaltered by the above transformation, it follows that the cusps are transformed into cusps, while the flexes of the x-y arches do not go into flexes on the petals, and an examination of the figures given shows that the flexes on the petals are closer to the outer cusps than to the inner ones.

Since $\tan \psi = \rho d\theta/d\rho = -(2/pk)(1+k \cos t)\sin t$, and this is zero for $t=0$, π, 2π, etc., it follows that the curve is tangent to the radii vectores for these values which are the same as the values of t giving the cusps; hence the cusp tangents are radii of the concentric circles $\rho=1+k$ and $\rho=1-k$ on which the outer and inner rows of cusps lie.

Equations (4), even with the limitations on the values of the arbitrary constants p and k, allow us to vary the tracery very considerably, the number of petals being at our disposal as well as the depth of the circular ring which the petals occupy, and we give in Figs. 2 and 3 two series of curves in the first

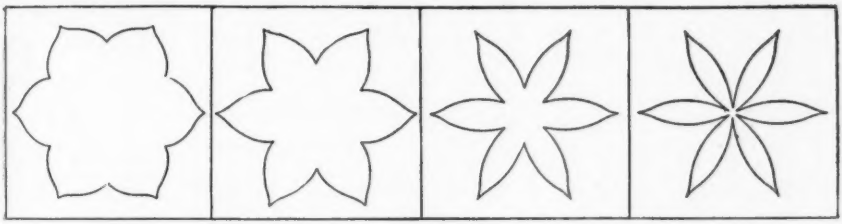


FIG. 3

of which the number of petals is varied while the depth of the ring is kept constant, and in the second of which the depth of the ring is varied whilst the number of petals is kept constant.

We may secure a further variation in form by replacing the integer p by a rational fraction, the equations now become

$$\theta = (c/p)(t - \tfrac{1}{2}\sin 2t), \qquad \rho = 1 + k\cos t, \tag{5}$$

where c and p are both integers and c is less than and prime to p. ρ now goes through its entire variation, and hence one petal is completed as t varies from 0 to 2π, or from 2π to 4π, etc., whilst for this same increase in the parameter t the angle θ increases by the quantity $(c/p)2\pi$ which is therefore the angular measure of the sector in which each petal is contained; hence in the entire circuit of 360° there will be p/c petals, but as p/c is not an integer the last petal will be incomplete and the curve will not close in at the first revolution of the radius vector but only when c complete revolutions have been made. Thus we will have a more complicated tracery consisting of p petals formed in

c complete circuits of the circle as shown in Figs. 4 and 5, for which c/p equals respectively 2/3 and 2/5. The greater the values of c and p the more complicated the figure becomes and if we were to allow, as a further extension, c/p to take on irrational values we would have an endless curve with an infinite number of petals filling the entire circular ring bounded by the two concentric circles of cusps. These forms are of course impossible in actual architectural designs; and in fact there are few if any examples of forms corresponding strictly to fractional values of p/c though we find pseudo examples of such forms constituted by two interlacing curves of the simpler form represented by equations (4).

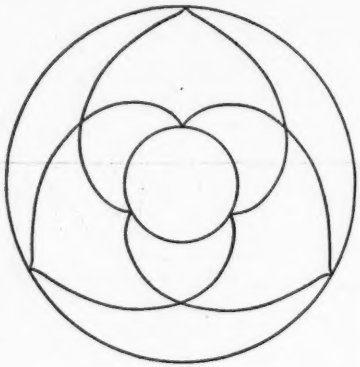


FIG. 4

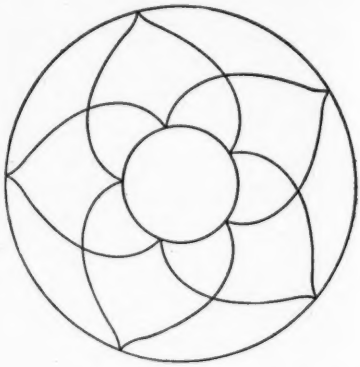


FIG. 5

Returning now to these simpler forms we find that the artistic character of the petals depends considerably on the position of the flex; if we compare the cuspidal arches of the x-y curve (Fig. 1) with the corresponding petals of the θ-ρ curve (Fig. 2) into which they are transformed we find that the former are more symmetrical (though not more graceful) than the latter inasmuch as the flex in the arches is midway between the cusps as measured either along the length of the curve or in the vertical direction between the cusps, while in the petals the flexes are much nearer the outer cusps than the inner ones. The position of the flex in flowing tracery differs considerably in the different examples which we have examined in Gothic structures, and hence it is well to have a means of modifying the shape of the petals more completely than the simple equations already discussed allow us to do. For this purpose we add a new term to the parametric value of ρ which will leave the position of the cusps unaltered but will bring the flexes nearer to the inner cusps. This added function of t should therefore become zero whenever the original petal has a cusp, that is for $t=0$, π, 2π, etc.; any function containing $\sin t$ as a factor will satisfy this condition but if we wish further to retain the symmetry of the petal

with respect to its central line or cusp tangent we must choose a function that will have the same value for $2n\pi+\varphi$ as for $2n\pi-\varphi$, where φ is any angle whatever; $\sin^2 t$ is a function of this character but we prefer to keep the central point of the portion of the petal joining the inner and outer cusps also unaltered and this leads us to $m \sin^2 t \cos t$ for the form of the added term and our modified equations are

$$\theta = (c/p)(t - \tfrac{1}{2}\sin 2t), \quad \rho = 1 + k\cos t - m\sin^2 t \cos t, \tag{6}$$

where m is an arbitrary constant determining the direction and amount of the bending. The curves given in Fig. 6 show the effect of this new term on the six petal design.[1] The first curve (a) is the fundamental or unmodified curve

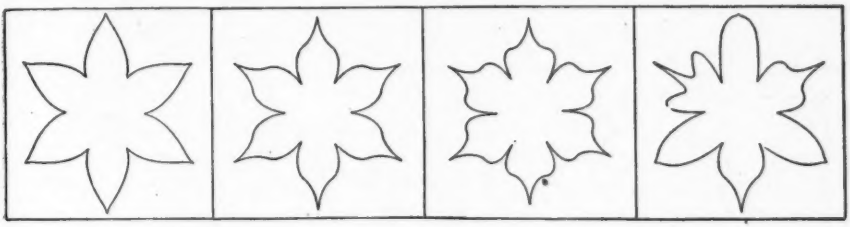


FIG. 6

given by equations (4); the second and third figures (b and c) are complete curves for which m equals respectively 1/4 and 1/2, and the fourth curve is a composite one in which the petals are taken from different curves with values of m varying from 1 to $-\frac{1}{2}$.

So far we have given the equations which represent the curves of pure flowing tracery; to complete the consideration of our subject we shall touch briefly on the flamboyant style. In this style the artist forsook the radial symmetry of the figure and began, apparently for the sake of securing novelty in his designs, to twist the figure spirally as can be seen in a number of the buildings erected during the period when Gothic architecture was declining from the purity and perfection it had attained in the thirteenth century. In order to introduce this twisting of the figure into our equations we merely make a transformation which will change radial straight lines into spirals through the origin. The equation of a radial straight line is $\theta = a$, where a is any constant, whilst the equation $\theta = a + b\rho$, where b is an arbitrary constant and ρ is the

[1] At the Kansas City meeting an informal vote was taken on the question "Which of the curves in Fig. 6 is the most graceful?" It would be interesting to know what is the opinion in this matter of a larger circle of individuals than were gathered at the meeting, and the author would be grateful to any reader of this article who would inform him by postal which one of these curves pleases him or her the most. (Address: Georgetown University, Washington, D. C.)

distance from the origin to the point (θ,ρ), gives us a spiral. Hence to give our curves the spiral twisting we use the equations

$$\theta = (c/p)(t - \tfrac{1}{2}\sin 2t) + b\rho, \quad \rho = 1 + k\cos t. \tag{7}$$

The value of b determines the amount of spiral twisting and Fig. 7 shows a couple of the flamboyant curves with the corresponding values of b indicated under each of them.

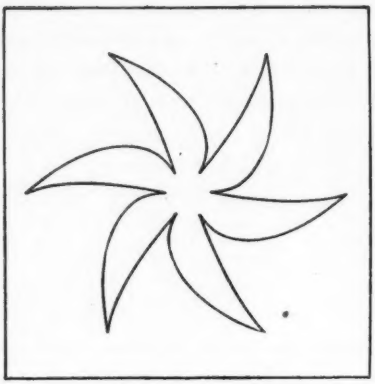


FIG. 7a. $b=0.087$

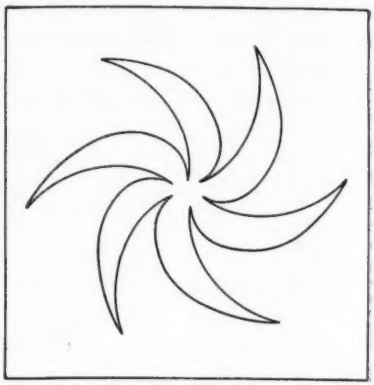


FIG. 7b. $b=0.174$

These curves could also be subjected to the various modifications introduced above in the case of the flowing traceries, but it seems hardly worth while pursuing the subject further in this direction as the flamboyant style is almost universally admitted to be a decadent form of Gothic; its fiery lines may indeed attract the attention of the onlooker but it lacks that perfection of proportion and harmony which is so characteristic of flowing Gothic and gives it its enduring charm.

ON THE DE LONGCHAMPS CIRCLE OF THE TRIANGLE

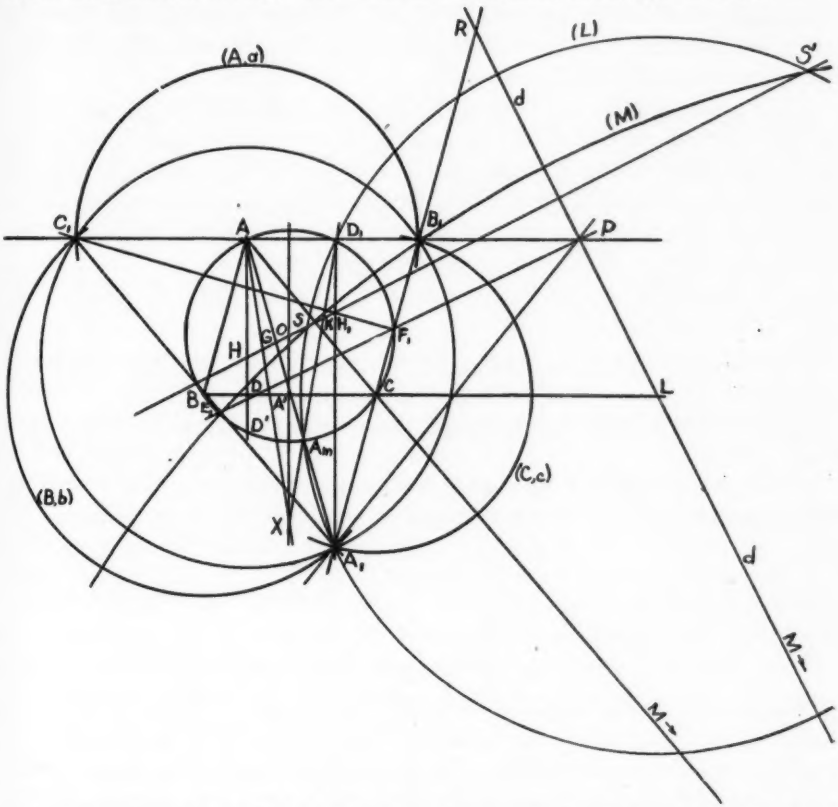
By NATHAN ALTSHILLER-COURT, University of Oklahoma

Introduction. The three circles (A,a), (B,b), (C,c) having for centers the vertices A, B, C of a given triangle ABC, and for radii the corresponding opposite sides were first considered by G. de Longchamps in an article "Sur un nouveau cercle remarquable du plan du triangle" published in the *Journal de Mathématiques Spéciales* in 1886, pp. 57 etc. In this analytical study the author was chiefly interested in the orthogonal circle of these three circles, and in the radical axis of this orthogonal circle with the circumcircle of the

basic triangle. This radical axis, the orthogonal circle, and the center of this circle are referred to respectively as the "de Longchamps axis, circle, and point" of the triangle.

In 1891, E. Vigarié gave some properties of the radical circles of the circles (A,a), (B,b), (C,c) in the *Journal de Mathématiques élémentaires*, pp. 63 etc.

E. Lemoine calculated the radii of the two circles which touch the circles (A,a), (B,b), (C,c) all internally and all externally (*Journal de Mathématiques élémentaires*, 1895, p. 139. See also *Mathesis*, 1906, pp. 59 etc.[1])



In the present article the bisecting circles and the circles of similitude of the circles (A,a), (B,b), (C,c) are considered. The results of the sections 1, 2, 3, 9*b* are known. For the convenience of the reader it was deemed advisable to reproduce them here.

[1] For bibliographical references to some related topics see M. Simon, *Über die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert*, p. 104, Leipzig, Teubner, 1906.

1. The line of centers $BC=a$ of the two circles (B,b), (C,c) being smaller than the sum and greater than the difference of the radii b, c, the two circles intersect in two real points A_1, D_1; let A_1 denote the point lying on the opposite side of BC from the vertex A.

The two pairs of opposite sides of the quadrilateral ABA_1C are equal, hence A_1 is the point of intersection of the parallels to the sides AB, AC of ABC drawn through the opposite vertices C, B.

The triangles ABC, D_1BC are congruent as having their sides respectively equal, hence AD_1 is parallel to BC, and therefore perpendicular to D_1A_1, since the common chord is perpendicular to the line of centers BC. Similarly for the analogous points B_1, E_1; C_1, F_1.

The points A_1, B_1, C_1 are thus the vertices of the triangle formed by the parallels to the sides of ABC through the opposite vertices, *i. e.*, $A_1B_1C_1$ is the anticomplementary triangle of ABC, and $D_1E_1F_1$ is the orthic triangle of $A_1B_1C_1$. Thus: *The three circles* (A,a), (B,b), (C,c) *have for diameters the sides of the anticomplementary triangle of the basic triangle, and for common chords the altitudes of the anticomplementary triangle.*

The reader may notice that the points D_1, E_1, F_1, lie on the circumcircle (O) of the basic triangle ABC.

2. The altitudes A_1D_1, B_1E_1, C_1F_1 of $A_1B_1C_1$ being the radical axes of the circles (A,a), (B,b), (C,c) taken in pairs, the orthocenter H_1 of $A_1B_1C_1$ is therefore the radical center of these three circles. *i. e.*, the de Longchamps point of ABC.

The two triangles $A_1B_1C_1$ and ABC being homothetic with respect to their common centroid G (the homothetic ratio being -2), we have

$$GH_1 : GH = -2,$$

where H is the orthocenter of ABC. On the other hand O being the circumcenter of ABC we have

$$GH : GO = -1, \qquad \text{hence} \qquad OH_1 : OH = -1.$$

Consequently: *The de Longchamps point of a triangle*: (a) *lies on the Euler line of the triangle*; (b) *is the symmetric of the orthocenter of the triangle with respect to its circumcenter*; (c) *is the orthocenter of the anticomplementary triangle.*

3. The power of H_1 with respect to the circles (A,a), (B,b), (C,c) is respectively $H_1A_1 \cdot H_1D_1 = H_1B_1 \cdot H_1E_1 = H_1C_1 \cdot H_1F_1$, hence the center H_1 as well as the radius of the de Longchamps circle (H_1) of ABC is identical with the corresponding elements of the conjugate circle of $A_1B_1C_1$, hence the two circles are identical.

Now the conjugate circle of $A_1B_1C_1$ is coaxal with its circumcircle (O_1) and its nine point circle,[1] the latter being identical with the circumcircle (O)

of ABC. We have therefore: (a) *The de Longchamps circle of a triangle is the conjugate circle of the anticomplementary triangle.* (b) *The de Longchamps axis of a triangle is the radical axis of its circumcircle with the circumcircle of the anticomplementary triangle.*

4. For the foot D of the altitude AD of ABC we have

$$AB^2-BD^2=AC^2-CD^2$$

or

$$CD^2-BD^2=AC^2-AB^2=b^2-c^2.$$

Similarly for the feet E, F of the altitudes BE, CF. Hence:[2] *The altitudes of the basic triangle are the loci of the centers of the circles which bisect (i. e., cut in diametrically opposite points) the three circles (A,a), (B,b), (C,c) taken in pairs.*

5. The point H common to the three loci AD, BE, CF (4)[3] is the center of a circle bisecting the three circles (A,a), (B,b), (C,c). It is readily seen that this bisecting circle is identical with the circumcircle (O_1) of $A_1B_1C_1$.

The second point of intersection D' of AD with the circumcircle (O) of ABC is the symmetric of H with respect to the side BC, hence the circle (D') described with D' as center and bisecting the two circles (B,b), (C,c) is the symmetric of (O_1) with respect to BC. Similarly for the analogous circles (E'), (F').

It follows that the lines AB, AC are the radical axes of (O_1) with the circles (E'), (F') respectively, hence the radical axis of (E') and (F') is the perpendicular from A upon the line $E'F'$, *i. e.*, this radical axis is the radius OA of the circumcircle (O) of ABC.[4]

The line of centers OD' of the two circles (O), (D') is equal to the radius of (O), and since the radius of (D') is double the radius of (O), the two circles are tangent at the diametric opposite D_1 (1) of D' on (O). To sum up: *The circles (A,a), (B,b), (C,c) taken in pairs are bisected by the circles having for centers the second points of intersection of the altitudes of the basic triangle with its circumcircle, and for radii the diameter of this circumcircle.*

These three circles have for their radical center the center of the circumcircle and are tangent to this circle, the points of contact being the vertices of the orthic triangle of the anticomplementary triangle.

6. The line BC is the radical axis of the circles (O_1) and (D') (4, 5), and the common tangent D_1L at D_1 is the radical axis of the circles (O) and (D'),

---

[1] Nathan Altshiller-Court, *College Geometry*, Johnson Publishing Company, Richmond, Va., 1925, p. 217.

[2] *Ibid.*, p. 168.

[3] A number in parentheses refers to the corresponding section.

[4] *Ibid.*, p. 85.

hence the trace L of D_1L on BC is a point on the radical axis of the circles (O), (O_1), *i. e.*, on the de Longchamps axis of the triangle ABC (3b). Consequently: *The common tangents to the circumcircle of the basic triangle and the three bisecting circles* (5) *meet the corresponding sides of the basic triangle on the de Longchamps axis of this triangle.*

7. The lines B_1C_1 and E_1F_1 are the radical axes of the circle (A,a) with the circles (O_1) and (O) respectively, hence the point $P\equiv(B_1C_1,\ E_1F_1)$ is a point on the radical axis of the circles (O) and (O_1), *i. e.*, on the de Longchamps axis of the triangle ABC (3b). Thus P is the point of intersection of the radical axes d and B_1C_1 of the circle (O_1) with the circles (H_1) (3) and (A,a) respectively, hence P lies on the radical axis of (H_1) and (A,a).

The four points B_1, C_1, E_1, F_1 determine a complete quadrangle inscribed in the circle (A,a), hence the diagonal points $A_1\equiv(B_1F_1,\ C_1E_1)$, $H_1\equiv(B_1E_1,\ C_1F_1)$, $P\equiv(B_1C_1,\ E_1F_1)$ determine a triangle A_1H_1P conjugate with respect to (A,a), and therefore A is the orthocenter of A_1H_1P. Thus the line A_1P is perpendicular to the line of centers AH_1 of the circles (A,a), (H_1) and passes through the point P of their radical axis, hence A_1P coincides with the radical axis of these two circles. Consequently: *The lines joining the vertices of the anticomplementary triangle to the traces of the de Longchamps axis on the opposite sides of this triangle are the radical axes of the de Longchamps circle with the circles* (A,a), (B,b), (C,c).

8. The radical axes A_1P, B_1Q, C_1R of (H_1) with the circles (A,a), (B,b), (C,c) (7) form a triangle whose sides meet the sides B_1C_1, C_1A_1, A_1B_1 of $A_1B_1C_1$ in the points P, Q, R, of the de Longchamps axis d of ABC (7). On the other hand the point of intersection of the radical axes B_1Q, C_1R of (H_1) with (B,b), (C,c) respectively lies on the radical axis $A_1H_1D_1$ of the last two circles, hence: *The anticomplementary triangle and the triangle formed by the radical axes of the de Longchamps circle with the circles* (A,a), (B,b), (C,c) *are homological, the center and axis of homology being the de Longchamps point and the de Longchamps axis of the basic triangle.*

9. The line A_1A joining the vertex A_1 of the triangle A_1H_1P to its orthocenter A (7) is perpendicular to PH_1. On the other hand the polar of P with respect to (H_1) passes through the pole A_1 of B_1C_1 with respect to (H_1) and is perpendicular to the line PH_1 joining P to the center H_1 of (H_1), hence this polar coincides with A_1A. Consequently: (a) *The medians of the basic triangle are the polars, with respect to the de Longchamps circle, of the traces of the de Longchamps axis on the sides of the anticomplementary triangle.* (b) *The centroid of the basic triangle is the pole of the de Longchamps axis with respect to the de Longchamps circle of the triangle.*

10. Let (L), (M), (N) be the circles of similitude of the circle (A,a), (B,b), (C,c) taken in pairs. The circle of similitude (L) of the circles (B,b), (C,c) is coaxal with these circles. Then the three circles of similitude (L) (M), (N) are orthogonal to the circles (O), (O_1), (H_1),[1] and therefore coaxal, their radical axis being the line of centers OH_1 of the circles (O), (H_1), *i. e.*, the Euler line of ABC. The centers L, M, N, of (L), (M), (N) are the points of intersection of the de Longchamps axis with the sides of the basic triangle ABC (6).

The circles (M), (N) pass through the points B_1, C_1, respectively. Since they are orthogonal to (O_1), they are tangent to the radii HB_1, HC_1 of (O_1). Now the point H lies on the radical axis of (M), (N), hence B_1, C_1, are anti-homologous points on these two circles,[2] and the line B_1C_1 meets the line of centers $d \equiv MN$ in a center of similitude of (M), (N), *i. e.*, this center of similitude coincides with P (7) and lies on the radical axis PA of the two circles (A,a), (H_1).

The line AP is the harmonic conjugate, with respect to AB, AC, of the line AA' joining A to the mid-point A' of BC, hence the line AA' meets d in the harmonic conjugate of P with respect to the centers M, N of the circles (M), (N), *i. e.*, in the second center of similitude of these two circles. Consequently: *The three circles of similitude of the three circles* (A,a), (B,b), (C,c) *taken in pairs have for their centers of similitude the traces of the de Longchamps axis on the medians of the basic triangle and on the three radical axes of the de Longchamps circle with the circles* (A,a), (B,b), (C,c).

These three pairs of centers of similitude are pairs of conjugate points with respect to the de Longchamps circle (9a).

11. The centers of similitude U, U' of the two circles (B,b), (C,c) divide the line of centers BC harmonically in the ratio $b:c$, hence these points are the traces on BC of the bisectors of the angle D_1 of the triangle D_1BC. Now the circle of similitude (L) of the circles (B,b), (C,c) has for diameter the segment UU' and passes through the point D_1, hence (L) is the circle of Apollonius of D_1BC corresponding to the vertex D_1. On the other hand the triangle D_1BC is the symmetric of the basic triangle ABC with respect to the circumdiameter of ABC perpendicular to BC. Similarly for the circles (M), (N). Therefore: *The three circles of similitude of the circles* (A,a), (B,b), (C,c) *are the symmetrics of the circles of Apollonius of the basic triangle with respect to the perpendicular bisectors of the corresponding sides of this triangle.*

12. The triangle A_1BC is the symmetric of the triangle D_1BC with respect to BC, and since the circle (L) (11) is its own symmetric with respect to BC,

[1] J. L. Coolidge, *A Treatise on the Circle and the Sphere*, Oxford, 1916, p. 107.

[2] *College Geometry*, p. 166.

(L) is also the Apollonian circle of A_1BC passing through A_1. Consequently the lines A_1U, A_1U' are the two bisectors of the angle BA_1C. But the angle BA_1C is identical with the angle $B_1A_1C_1$, hence: *The sides of the basic triangle are met by the bisectors of the corresponding angles of the anticomplementary triangle in the centers of similitude of the circles* (A,a), (B,b), (C,c) *taken in pairs.*

13. The triangles A_1BC and $A_1B_1C_1$ are homothetic, the point A_1 being the homothetic center and the ratio being $1/2$, hence the Apollonian circle (L) of A_1BC (12) corresponds in this similitude to the Apollonian circle (A_1) of $A_1B_1C_1$ passing through A_1. Thus the midpoint of the segment intercepted by (A_1) on the symmedian of $A_1B_1C_1$ passing through A_1 lies on (L). But this midpoint is a vertex of the second Brocard triangle of $A_1B_1C_1$,[1] hence: *The circles of similitude of the three circles* (A,a), (B,b), (C,c) *taken in pairs pass through the respective vertices of the second Brocard triangle of the anticomplementary triangle.*

14. Let A_m be the second point of intersection of the median AA' of the triangle ABC with its circumcircle (O). The areas of the two triangles BAA_m, CAA_m having the common base AA_m, are to each other as the segments BA', CA', hence these areas are equal, and since the angles ABA_m, ACA_m are supplementary, we have

$$AB \cdot BA_m = AC \cdot CA_m, \quad \text{or} \quad A_mB : A_mC = b : c,$$

hence the point A_m belongs to the circle of similitude (L) of (B,b), (C,c). Similarly for the analogous points B_m, C_m. Thus: *The circles of similitude of the circles* (A,a), (B,b), (C,c) *taken in pairs pass through the second points of intersection of the circumcircle of the basic triangle with the corresponding medians of this triangle.*

15. The circle of Apollonius (L) of D_1BC is orthogonal to the circumcircle (O) of this triangle, hence their common chord D_1A_m (14) is the polar of the center L of (L) with respect to (O), and since L lies on BC, the polar D_1A_m will pass through the pole of BC with respect to (O), *i. e.*, through the point of intersection X of the tangents BX, CX to (O) at the points B, C. On the other hand D_1BC being the symmetric of ABC with respect to the diameter $OA'X$ of (O), the line D_1A_m X is the symmetric, with respect to $OA'X$, of the common chord of (O) and the Apollonian circle of ABC passing through A (11), *i. e.*, the symmedian AX of ABC.[2] Similarly for the analogous lines E_1B_mY, F_1C_mZ.

---

[1] *College Geometry*, p. 247.

[2] *College Geometry*, p. 236.

The diameter OX of (O) is the bisector of the angle $BXC \equiv ZXY$, hence the line $XA_\infty D_1$ is the isogonal conjugate of the line XA in the tangential triangle XYZ of ABC. Since the symmedians XA, BY, CZ of ABC meet in the Lemoine point of ABC, (which point is the Gergonne point of the triangle XYZ), therefore their isogonal conjugates XD_1, YE_1, ZF_1, are also concurrent in a point K'. Now the poles of these lines with respect to (O) are the centers L, M, N, of the circles of similitude (L), (M), (N), hence K' is the pole of the line $d \equiv LMN$ with respect to (O) and lies on the line OH. Thus :*The isogonal conjugate of the symmedian point of a triangle with respect to the tangential triangle lies on the Euler line of the basic triangle and is the pole of the de Longchamps axis of this triangle with respect to its circumcircle.*

16. The de Longchamps circle (H_1) of ABC is the conjugate circle of the triangle $A_1B_1C_1$ (3), hence (H_1) is real or imaginary according to whether $A_1B_1C_1$ is obtuse angled or acute angled, and since the triangles $A_1B_1C_1$ and ABC are similar, the triangle ABC may be substituted for $A_1B_1C_1$ in this condition. The preceding considerations, however, remain valid, as far as the circle (H_1) is involved, in either case. This is due to the fact that no use has been made of any point on the circle (H_1), while the properties dealt with were those of poles and polars, and of the radical axes. These properties depend upon the square of the radius of (H_1), a quantity which is real whatever the shape of ABC may be.[1]

Let us now assume that ABC is acute angled. In this case the circles (O), (O_1), (H_1) form a non-intersecting system of coaxal circles, and therefore the circles (L), (M), (N) of the conjugate coaxal system have two real points S, S' in common on their common radical axis OH_1, *i. e.*, on the Euler line of the triangle ABC. The points S, S' are the limiting points of the first coaxal system and therefore conjugate with respect to every circle of this system, and also symmetric with respect to the trace of the line $d \equiv LMN$ on OH_1. Consequently: *In the plane of an acute angled triangle there are two, and only two points each of which has the property that its distances from the vertices of the triangle are proportional to the sides of the triangle opposite the vertices considered. The two points lie on the Euler line of the triangle, are conjugate with respect to its circumcircle, and are symmetric with respect to the trace of the de Longchamps axis on the Euler line.*[2]

---

[1] M. Chasles, *Traité de Géométrie Supérieure*, second edition, pp. 502 etc., Gauthier-Villars, Paris, 1880.

[2] *Cf.* R. Lachlan, *Modern Pure Geometry*, p. 192, ex. 5, Macmillan, 1893.

QUESTIONS AND DISCUSSIONS

EDITED BY TOMLINSON FORT, Hunter College, Park Ave. and 68th St., New York, N. Y., and H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. MATHEMATICAL INDUCTION

By G. E. RAYNOR, Wesleyan University

In the MONTHLY for November 1920, Professor E. T. Bell has criticized the principle of mathematical induction from a logical point of view. His paper is preceded by some remarks by the editor and, in the April 1922 number, Mr. R. S. Hoar and the editor have discussed it further. However none of these gentlemen have attempted to give a specific reply to Professor Bell's closing sentence in which he asks in regard to mathematical induction " . . . . where is either a proof of it or its explicit statement as a postulate of logic to be found?" It is the purpose of this note to attempt to answer the above question and perhaps to reopen the discussion of mathematical induction.

Bertrand Russell in his *Introduction to Mathematical Philosophy*, published in 1919, in the chapter devoted to mathematical induction, states rather dogmatically that this principle must be taken as a definition. However, he completely ignores a paper published in 1913 by A. Padoa in the *Proceedings of the Fifth International Congress of Mathematicians*, Vol. II, in which Padoa shows that the principle may play other rôles in the deductive science of arithmetic. He reminds us that in any deductive science each proposition must have one of three rôles, it must be either a postulate or a definition or a theorem. He then outlines various systems in which the principle has played a part, with reference to works in which complete discussions may be found.

In Peano's *Arithmetices principia nova methodo exposita* published in 1899, mathematical induction plays the part of a postulate along with four others. In Russell's *The Principles of Mathematics* and later in Russell and Whitehead's *Principia Mathematica*, and also in the more recent work by Russell referred to above, it enters explicitly in the definition of inductive numbers. In 1900, Padoa succeeded in reducing Peano's five postulates to four, one of which, although not the principle of mathematical induction, is very nearly it. In

1908, M. Pieri gave four postulates from which the principle of mathematical induction as well as the rest of arithmetic may be deduced. Finally, in the paper which we are partially outlining, Padoa gives another set of postulates and definitions for arithmetic from which mathematical induction follows as a theorem.

It thus appears from Padoa's work and from that of Peano and Russell that the foundations of arithmetic may be so laid that the principle of mathematical induction may appear in the deductive science which arises from them either as a postulate, as a definition, or as a theorem.

It may also be mentioned that in his *Theory of Functions of a Real Variable* 2nd edition, vol. I, E. W. Hobson has given a proof of mathematical induction which is open to objection. His proof is based on a so-called definition of a "simply infinite ascending aggregate" but he does not show that the proof of the existence of such an aggregate is itself independent of the principle he is trying to prove.

II. A New Curve Connected with Two Classic Problems

By G. M. Juredini, Syracuse University

The curve is the locus of a point on a defined triangle which revolves about a circle in a particular manner. In Fig. 1, it is the path traced by the point P, which is the foot of the perpendicular let fall onto the hypotenuse of a right-angle triangle, RQX, the triangle revolving about a circle in such a manner that one of its sides, RQ, is always equal to and an extension of a radius of the circle, and the other side, QX, always reaches a fixed diameter of the circle, OX, as initial line, produced.

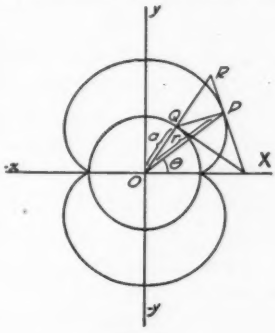


Fig. 1

The equation for the curve may be deduced as follows: In Fig. 1, we have angle $ROX=$ angle $ORX=\phi$. Let angle $POX=\theta$, radius of circle, or $RQ=a$, and line $PO=r$. Then, angle $RXO=180°-2\phi$, $OX=a\cdot\sec\phi$, $RP=a\cdot\cos\phi$, $PX=a(\sec\phi-\cos\phi)$, so that, in triangle PXO, we have $\sin(180°-2\phi)/r=\sin\theta/a(\sec\phi-\cos\phi)$, and from this it follows readily that

$$\sin\theta=2(a/r)(\sin^3\phi)\,. \tag{1}$$

Also, in triangle PRO, we have $r^2=4a^2+a^2\cos^2\phi-4a^2\cos^2\phi$, $\cos^2\phi=(4a^2-r^2)/3a^2$,

$$\sin^2\phi=1-(4a^2-r^2)/3a^2=(r^2-a^2)/3a^2. \tag{2}$$

From (1) and (2), we get

$$\sin\theta = 2a/r\left(\frac{r^2-a^2}{3a^2}\right)^{3/2}, \qquad \text{or} \qquad \sqrt[3]{y^2} = \frac{\sqrt[3]{4a^2}}{3a^2}(x^2+y^2-a^2) \tag{3}$$

which is the equation of the curve.[1] The graph takes the form indicated in Fig. 1.

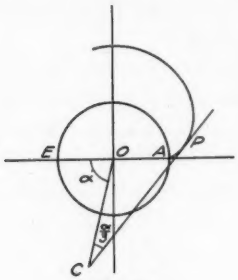


FIG. 2a

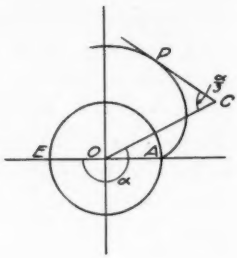


FIG. 2b

Application to Trisection of an Angle: In Fig. 2, let A be the starting point of the curve, and let EOC be the angle required to be trisected. Place OE on the side of the initial line that is opposite A, and measure off OC equal to diameter of circle. From C draw tangent to curve at P. Then PCO is one-third of EOC.

Application to Duplication, or Dimidiation of a Cube: In Fig. 1, let $QP=b$, and let P be at a distance $y=a$ from the line OX. From the right triangle QPR we have $\sin\varphi=(b/a)$, and when this value is inserted in (1) there results, after easy simplification,

$$2b^3=a^3. \tag{4}$$

---

[1] The curve of this note is a two-cusped epicycloid as may be verified by eliminating the parameter φ from the equations:

$$2x=a(3\cos\varphi-\cos 3\varphi), \qquad 2y=a(3\sin\varphi-\sin 3\varphi).$$

For such a curve, $(dy/dx)=\tan 2\varphi$, and the given construction is a consequence.

Thus the author has added a new curve to the galaxy of trisectrices already known. Concerning such curves Henri Brocard (*Notes de Bibliographie des Courbes Géométriques*, 1897, p. 290) says

"Trisectrices are infinite in number. One may mention: the quadratrix of Dinostratos, the spiral of Archimedes, the limaçon $r=1+2\cos\theta$, the conchoid of Nicomedes, the equilateral hyperbola, Maclaurin's trisectrix, and a host of others listed by Aubry, *Journal de Mathématiques Spéciales*, 1896, pp. 76–84, 106–112."

N. ANNING

III. Note on a Linear Diophantine Equation

By Elizabeth B. Cowley, Vassar College

1. One of the arithmetical puzzles that date back to mediaeval times is the measuring problem of the three vases, where it is required to divide into two equal parts the contents of an 8-ounce vase if the only empty vases hold 5 ounces and 3 ounces respectively. Various modifications of this specific form of the problem are to be found in collections of mathematical recreations. This problem, which was recently included in a set of psychological test questions, is also interesting because of its connection with certain linear Diophantine equations.

In the fifth edition[1] of Bachet's book the editor uses A, B, and C instead of particular numbers and notes that the various methods of solution are of two types. By one method the B-vase is filled directly from the A-vase and then the C-vase is filled and refilled from the B-vase (each time the C-vase is filled its contents are emptied into the A-vase), until the contents of the B-vase are $A/2$ or are less than C. In the latter case, the contents of the B-vase are emptied into the C-vase and then the B-vase is filled again from the A-vase. If after m fillings of the B-vase its contents are $A/2$, we have the equaton $mB-nC=A/2$, a condition which can always be satisfied, provided that B and C are prime to each other, or that $A/2$ contains all factors common to B and C. That condition is also sufficient if $A \not< (B+C)$. But if $A<(B+C)$, there may not always be enough in the A-vase to refill the B-vase. The editor cites two examples, 20,13,9 and 16,12,7 and says that the first is possible and the second not possible but that these facts are not known *a priori*.

By the second method the C-vase is filled and refilled from the A-vase and emptied into the B-vase until the B-vase is full. Then it is emptied into the A-vase and the remaining contents of the C-vase are turned into the B-vase. If, after m' fillings of the B-vase, the contents of the A vase are $A/2$, we have $A-n'C+m'B=A/2$ or $n'C-m'B=A/2$. The conditions for this method (the C method) are the same as those for the other method (the B-method).

I shall find the conditions that determine *a priori* whether a solution is possible if $A<(B+C)$ and I shall apply to this problem the graphical method of solution employed by C. A. Laisant[2] for certain types of equations.

These conditions are arranged in two tables, called the B-table and the C-table. In order to avoid trivial cases, the following assumptions are made:

---

[1] *Problèmes Plaisantes & Délectables qui se font par les nombres* par Claude-Gaspar Bachet sieur de Mézeriac, cinquième édition, revue, simplifiée et augmenteé par A. Labosne, Professeur de Mathématiques. Paris, 1884.

[2] *Association franç. av. sc.*, 1887, 218–235.

I. $A>B>C$ (where A, B, C are positive integers),
II. A is an even number,
III. $A/2 \neq nC$ (where n is an integer),
IV. B and C are prime to each other. (If B and C contain common factors which are in $A/2$, the problem is practically the repetition of a simpler problem; example, 16,10,6 and 8,5,3). Hence, $B=sC+R$, where $R<C$ and R and C are prime to each other.

2. Method of obtaining the B-table. The first filling of the B-vase is always possible, since $A>B$. Then the contents of the three vases are $A-B, B, 0$. Next they become $A-B+sC, 0, R$, or $A-R, 0, R$.

The second filling of the B-vase is impossible if $A-R<B$. If $A-R=B$, the B-vase can be filled but the A-vase is emptied in the process. In either case the combined contents of the other two vases is A. If we continue, we are really using the C-method. Hence we say that the B-method fails in the second attempt at filling the B-vase if $A-R \leqq B$. But if $A-R>B$, the contents of the three vases are $A-R-B, B, R$. The number of fillings of the C-vase to be had from the combined contents of the B and C vases is s if $R<C/2$ and it is $(s+1)$ if $R>C/2$. (It must be noted that $R \neq C/2$, since R and C are prime to each other). If $R<C/2$, the contents become $A-2R, 0, 2R$. Then the third filling is possible (without emptying the A-vase) if and only if $A-2R>B$. If $R>C/2$, the contents become $A-(2R-C), 0, 2R-C$; and the B-vase can always be filled without emptying the A-vase (since $A-B>R$ and $C>R$, then $A-B+C>2R$ or $A-2R+C>B$). Hence, the A-vase is emptied in the third filling of the B-vase when and only when $A-B \leqq 2R$ with $R<C/2$. This line of reasoning may be continued.

3. Method of obtaining the C-table. Since $B>sC$, the C-method will fail in the first attempt at filling the B-vase if $A \leq (s+1)C$. If $A>(s+1)C$, let $A=(s+1)C+r$, where r must be less than R. In this case the contents are $r, B, C-R$ and then $B+r, 0, C-R$. In order to obtain the second filling of the B-vase, we shall need s fillings of the C-vase if $R<C/2$. We can always get these fillings from the A-vase; then the contents are $R+r, B, C-2R$ and then $B+R+r, 0, C-2R$. But if $R>C/2$, we shall need $(s+1)$-fillings of the C-vase; and they can be obtained if and only if $B+r>(s+1)C$, *i.e.*, if $R+r>C$ Hence, the C-method fails in the second attempt to fill the B-vase if $R>C/2$ and $(R+r) \leq C$. This line of reasoning can be continued.

4. B-table. Conditions for failure at each filling of the B-vase:

I. The method never fails.
II. $A-R \leqq B$.
III. $A-2R \leqq B$ with $R<C/2$.
IV. $A-3R \leqq B$ with $R<C/3$,
$A-(3R-C) \leqq B$ with $C/3<R<2C/3$.
V. $A-4R \leqq B$ with $R<C/4$,
$A-(4R-C) \leqq B$ with $C/4<R<C/2$,
$A-(4R-2C) \leqq B$ with $C/2<R<3C/4$.

5. C-table. Conditions for failure at each filling of B-vase:

I. $A \leqq (s+1)C$.
II. $R+r \leqq C$ with $R>C/2$.
III. $2R+r \leqq C$ with $C/3<R<2C/3$,
$2R+r \leqq 2C$ with $2C/3<R$.
IV. $3R+r \leqq C$ with $C/4<R<2C/4$,
$3R+r \leqq 2C$ with $2C/4<R<3C/4$,
$3R+r \leqq 3C$ with $3C/4<R$.

V. $4R+r \leqq C$ with $C/5<R<2C/5$,
$4R+r \leqq 2C$ with $2C/5<R<3C/5$,
$4R+r \leqq 3C$ with $3C/5<R<4C/5$,
$4R+r \leqq 4C$ with $4C/5<R$.

6. Graphical methods of solution. The equations for the B-method and the C-method are respectively $mB-nC=A/2$ and $n'C-m'B=A/2$. If these are written

$$Cx' - Bz' = -A/2 \tag{1}$$

and

$$Cx - Bz = A/2, \tag{2}$$

they are examples of the types of equation $rx-pz=\pm a$ (r and p prime to each other and a and r less than p) which Laisant solved by a lattice of points. For equation (2), construct the points whose abscissas are $1,2,3, \ldots, B$ and whose ordinates are the corresponding residues, less than p, modulo p, of C, $2C, 3C, \ldots . BC$. These points are contained in a square of side B and resting on the x- and y- axes. In the square, draw the (C) line-segments of slope C, through the points. To solve the equation for known values of A, B, C, locate the point whose ordinate is $A/2$. Its abscissa equals x and its z equals the number of line segments to the left of the segment on which it lies. Laisant also has a method for equation (1); but I find that a slightly different method is preferable here. Since $x+x'=B$ and $z+z'=C$, then x' equals the distance of the point from the right side of the square and z' equals $1+$ the number of line-segments to the right of the segment on which the point lies.

The points with which we are especially concerned here lie within a horizontal band bounded by the lines $y=B/2$ and $y=(B+C)/2$. If a point of this band lies on or below the line $y=(s+1)C/2$, it fails by CI. Points on or below the line $y=(B+R)/2$ fail by BII. Hence points on or below both these lines fail by both the B and the C methods. Any point on the left-most line segment has its $A/2=kC$; while a point on the right-most line segment has its $A/2=B-lC$. Other points in the band must be further tested.

EXAMPLE: $B=61$, $C=24$. Points $A/2=31, 32, 33, 34, 35, 36$, fail by both methods. $A/2=37$ lies on the right-most line. Since $R>C/2$, points 38,39, 40, 41, 42 cannot fail by $BIII$. Point 39 succeeds by the B-method, since its $z'=3$. It is easy to find that point 41 succeeds and points 38, 40, and 42 fail. Trying the C-method, since $R>C/2$, values of $A/2$ for which $r \leqq 11$ fail by $C\,II$. This excludes all points except 42. This point succeeds by the C-method. In the cases cited by Labosne, 20, 13, 9 has $s=2$ and avoids CI and CII, and hence succeeds by the C-method, but fails by $BIII$; and 16, 12, 7 has $s=4$ and fails by CII and also fails by BII. Some other sets for which both methods fail are: 24, 19, 8; 34, 31, 10; 42, 36, 11; 44, 37, 10; 44, 35, 24.

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS.

Economic Statistics. By W. L. CRUM and A. C. PATTON. Chicago, A. W. Shaw Company, 1925. xii+493 pages. Price $4.00.

This book is divided into three parts. Part I deals with the sources, collection, tabulation, and charting of statistical data. Part II deals with general analytical methods by the use of averages, measures of dispersion, skewness, and correlation. Part III deals with the analysis of time series by the use of index numbers, link relatives, secular trend, seasonal variation, cyclical fluctuation and correlation of time series.

This book is designed to meet the needs of students who are interested in the application of modern statistical methods to economic problems. This purpose is carried out by drawing illustrative material almost entirely from economic sources, although many of the methods are applicable to other classes of data. This book, in common with several other books which have recently appeared, aims to introduce statistical processes to the student who has had but little mathematical training. It is assumed, however, that the reader of the book "has more than a reminiscence of the ground covered by elementary algebra." The most striking characteristic of the book is its richness in actual illustrative material. In this respect there is very fine evidence that the authors have followed class room and laboratory experience in preparing their material. The reviewer is in complete agreement with the authors in the view that numerical exercises should have a prominent place in the beginning statistics course.

The book may seem somewhat wordy to the mathematical reader. However, the reviewer believes it will be found very teachable to a certain large group of students of economics. The limitation of the book to economic statistics accounts for the relatively large amount of space given to time series and the relatively small amount to random sampling and partial and multiple correlation. The reviewer has a feeling that the idea of selection of "sample by design" (p. 205) might well have been elaborated.

The definition of correlation on p. 218 does not seem particularly useful. The genetic method of development adopted hardly required definition. Indeed it is not clear that the definition has been used. As stated on p. 211 the definitions of positive and negative skewness are opposite to the definitions of Karl Pearson. The authors follow Charlier instead of Pearson in this convention. The reviewer approves of the change because he has found in practically all his classes that students feel the Pearson convention is an unnatural one.

The devices used in the analysis of time series are empirical to a high degree, and there is considerable difference of opinion on certain points such as the measurement of seasonal variation by the median-link-relative method. While it is to be hoped that improvements will be effected in these devices, those presented in the book are important tools in the quantitative description of time series.

On p. 297, line 18 from bottom, p_i/q_0 should read p_iq_0.

H. L. RIETZ.

Gli Elementi d'Euclide e la Critica Antica e Moderna. Edited by Federigo Enriques col concorso di diversi collaboratori. Libri I–IV. Rome, A. Stock, 1925. 325 pages. Price 25 lire.

This is the first volume of a series "Per la Storia e la Filosofia delle Matematiche," published under the auspices of the Istituto Nazionale per la Storia delle Scienze Fisiche e Matematiche, and edited by Professor Enriques of the University of Bologna. It is the outcome of a suggestion made by certain leaders in the training of teachers of mathematics and it proceeds upon the principle that it is only by a historic survey of the classical textbooks of geometry that a sound basis can be laid for a critical study of the subject. The result of the abandoning of Euclid as a text has, in the opinion of Professor Enriques, been a serious loss in the equipment of the teacher.

He begins his work with a brief survey of the history of ancient geometry, followed by a bibliography of some of the leading editions of the *Elements*. He then gives the text of Euclid, in Italian translation, based upon the Heiberg edition. In the translation and editorial work he was assisted by Dr. Maria Teresa Zapelloni. As in the case of the Heath edition, the definitions and propositions are followed by critical notes containing much historical material, the one-line definition of *point*, for example, having a full page of commentary, and similarly for *line*, *plane surface*, *angle* (3 pp.), and others. There is also an extended discussion of the postulates and axioms, and the commentary relating to the Pythagorean Theorem (I, 47) fills six pages.

It will therefore be seen that, with such an editor as Professor Enriques to assure the scholarship of the commentary, the work will be helpful to any student of the subject and to any reader of the most influential textbook on mathematics ever written. It is significant that the teachers of Italy, which now ranks among the half dozen leading nations in mathematical activity, should feel the need of such an edition of Euclid at a time when some of our American educators are proclaiming the uselessness and indeed the happy death of the science of geometry.

DAVID EUGENE SMITH.

Matematiche, Scienze Naturali e Medicina nell' Antichita Classica. By J. L. HEIBERG, Traduzione di Gino Castelnuovo. Rome, A. Stock, 1924. 188 pages. Price 12.60 lire.

This is the second volume of the series "Per la Storia e la Filosofia delle Matematiche," published under the editorial direction of Professor Enriques of Rome. It is a translation of Professor Heiberg's *Naturwissenschaften und Mathematik im Klassischen Altertum*, Leipzig, 1912. Besides the translation there are numerous footnotes, largely referring to articles or comments by Professor Enriques, or to notes in Dr. Singer's English edition. There is a brief bibliography,—too brief for those needing generous assistance.

Heiberg's work is so well known to historians of mathematics that no extended description is desirable. It is a general essay rather than a detailed history, but it has scientific value because of the fact that the author stands out as probably the best-versed scholar in the field of Greek mathematical texts of any man of modern times. The topics considered are as follows: (1)Ionic natural philosophy; (2) The Pythagoreans; (3) Medicine in the 5th century B.C.; Hippocrates; (4) Mathematics in the 5th century B.C.; (5) Plato and the Academy; (6) Aristotle; the Peripatetics; (7) The Alexandrian School; (8) The Epigones [successors; that is, later Greek writers]; (9) The Romans; (10) Greek scientific literature of the Byzantine period.

DAVID EUGENE SMITH.

ARTICLES IN CURRENT PERIODICALS

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

Proceedings of the National Academy of Sciences, volume 12, no. 4, April 1926: "On Laguerre's Series" by E. Hille, 261-268; "Groups containing a relatively small number of Sylow subgroups" by G. A. Miller, 269–273. Volume 12, no. 5, May, 1926: "On Laguerre's Series, Third Note" by E. Hille, 348–351; "On conformal geometry" by T. Y. Thomas, 352–358; "Concerning indecomposable continua and continua which contain no subsets that separate the plane" by R. L. Moore, 359–363.

Transactions of the American Mathematical Society, volume 28, no. 1, January, 1926: "Intersections and transformations of complexes and manifolds" by S. Lefschetz, 1–49; "Divergent double sequences and series" by G. M. Robison, 50–73; "On certain families of orbits with arbitrary masses in the problem of three bodies" by F. H. Murray, 74–118; "Existence theorems for a linear partial difference equation of the intermediate type" by C. R. Adams, 119–128; "An algebra of sequences of functions, with an application to the Bernoullian functions" by E. T. Bell, 129–148; "Bundles and pencils of nets on a surface" by E. P. Lane, 149–167; "On the momental constants of a summable function" by R. E. Langer, 168–182; "Fundamental systems of formal modular protomorphs of binary forms" by W. L. G. Williams, 183-197; "Non-synchronized relative invariant integrals" by K. P. Williams, 198–206.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in solution of such problems.

3206. Proposed by D. H. Lehmer, University of California.

Prove the following theorems and show how they may be used in finding the factors of R:

THEOREM 1. Let R be a non-square integer of the form $8n+k$ and let $2^p(2m+1)$ be any even denominator of a complete quotient occurring in the expansion of $\sqrt{R}$ in a continued fraction, then, if $k=1$, $p\geqq 3$; if $k=4$ or 0, $p\geqq 2$; if $k=5$, $p=2$; if $k=2$, 3, 6, or 7, $p=1$.

THEOREM 2. If R contains a square factor, k^2, then every multiple of k appearing as a denominator of a complete quotient must also contain a factor k^2.

3207. Proposed by C. N. Mills, Normal, Illinois.

Prove that $\frac{1}{4}m^2$ is the maximum area of a triangle which can be formed with the lines a, b, c, subject to the condition that $a^3+b^3+c^3=3m^3$.

3208. Proposed by V. M. Spunar, Chicago, Illinois.

Disprove the following: If a,n are any integers and $a^x\equiv 1 \pmod{n}$ for $x=n-1$ but not when x is an aliquot part of $n-1$, the integer n is a prime. (Lucas, *Theory of Numbers*, I, p. 441.)

3209. Proposed by M. Zametkin, Jamaica, N. Y.

Given $4x^3-21x+14=0$, show that, if $A=10°/7$, $x_1=\sin 41A+\sin 37A+\sin 25A+\sin 13A-\sin 19A-\sin 11A$. Determine the other two roots, x_2 and x_3, in like sums of sines.

3210. Proposed by Thurmar Andrew, University of North Dakota.

Find the general solution of the probability of throwing any number with any number of dice. Dice are here taken to be polyhedrons, with any number, k, of faces numbered consecutively from 1 to k. (Assume all faces equally likely to turn up.)

3211. Proposed by J. A. Bullard, U. S. Naval Academy.

Find by integration the area of the ellipse $ax^2+2hxy+by^2+2gx+2fy+c=0$.

SOLUTIONS

2827 [1920, 186]. Proposed by B. F. Finkel, Drury College.

Find the equation of the envelope of the system of circles inscribed in a triangle having a given base and a given altitude.

I. SOLUTION BY OTTO DUNKEL, Washington University.

Let the rectangular axes be so taken that the extremities of the base are $(-c,0)$, $(c,0)$ and the ordinate of the variable vertex is h, then the coordinates (X,Y) of the center C of the circle satisfy the equation

$$X^2 = c^2 + \frac{hY^2}{2Y-h}. \qquad (1)$$

Let the circle C be tangent to the base, or the base extended if the circle is of the escribed kind, at M and to the envelope at $P(x,y)$, then P is the reflection of M on the tangent to (1) at C (see **2691**, *1919*, 131; also *1921*, p. 183). Let the inclination of the tangent be τ, then the figure shows that

$$y = 2Y\cos^2\tau, \qquad x = X - y\tan\tau. \tag{2}$$

Hence

$$y = \frac{2Y\cot^2\tau}{1+\cot^2\tau}, \qquad x = X - \frac{2Y\cot\tau}{1+\cot^2\tau}. \tag{3}$$

From (1) we can obtain $\cot\tau = dX/dY$ in terms of Y alone and also X in terms of Y. When these results are inserted in (3) there results the equations of the envelope in terms of the parameter Y. The elimination of this parameter will give the equation in x and y. These details are left to any reader to whom the results would be a source of pleasure or inspiration.

3118 [1925, 95, 522]. Proposed by Harry Langman, New York City.

If $n>2$ and ϵ is a primitive root of $\epsilon^n=1$, show that the determinant $|a_{ij}|$, of order $n-1$ and having the element a_{ij} equal to ϵ^{ij}, has the value

$$(-1)^{(n-1)(n-2)/4}n^{(n-2)/2}.$$

II. Solution[1] by Norman Anning, University of Michigan.

Call the determinant D and observe that, by bordering, D can be written

$$\begin{vmatrix} 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & \epsilon & \epsilon^2 & \cdot & \cdot & \cdot & \epsilon^{(n-1)} \\ 1 & \epsilon^2 & \epsilon^4 & \cdot & \cdot & \cdot & \epsilon^{2(n-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \epsilon^{(n-1)} & \cdot & \cdot & \cdot & \cdot & \epsilon^{(n-1)(n-1)} \end{vmatrix}.$$

We can build up the square of D in such a way that $D^2=|b_{ij}|$, $i,j=1,2,3,\cdots,n$, where the element $b_{ij}=1$ whenever either i or j is equal to 1 and, in all other cases,

$$b_{ij}=1+\epsilon^k+\epsilon^{2k}+\epsilon^{3k}+\cdots+\epsilon^{nk-k}, \qquad k=i+j-2.$$

Now, by well-known properties of primitive roots, this latter expression is equal to n when k is a multiple of n and is equal to 0 for all other values of k. Consequently

$$D^2=\begin{vmatrix} 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 \\ 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & n \\ 1 & 0 & 0 & \cdot & \cdot & \cdot & n & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & n & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & n & 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{vmatrix} = n^{n-2}\begin{vmatrix} n & 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 \\ 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \\ 1 & 0 & 0 & \cdot & \cdot & \cdot & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{vmatrix},$$

the second determinant being obtained from the first by multiplying the first column by n and then dividing all the rows except the first by n.

The coefficient of $n^{(n-2)}$ is a determinant of the nth order which we may call u_n. Then, if we subtract the second row from the first and expand in terms of the elements of the last column,

$$u_n=(-1)^n u_{n-1}.$$

Similarly,

$$u_{n-1}=(-1)^{n-1}u_{n-2}, \text{ etc.}$$

Finally,

$$u_4=(-1)^4u_3=(-1)^4\begin{vmatrix} 3 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}=-1.$$

It follows that $u_n=(-1)^{(n-1)(n-2)/2}$. Then $D^2=n^{(n-2)}(-1)^{(n-1)(n-2)/2}$, and the theorem follows, except for an ambiguity in sign.

---

[1] A solution by J. J. Nassau appeared in the Monthly for December, 1925.

Also solved by J. F. REILLY who pointed out that the statement of the problem is faulty since, in certain cases, the given determinant may have either of two values which are equal in magnitude but opposite in sign depending upon the choice of the primitive root e.

3138 [1925, 261]. Proposed by Nathan Altshiller-Court, University of Oklahoma.

The vertex of a triangle, whose base is fixed, moves along a straight line coplanar with the line of the base. Find (1) the locus of the orthocenter of the triangle; (2) the envelope of the line joining the feet of the two altitudes dropped from the two fixed vertices of the triangle. See solution of **3152** below.

3152 [1925, 481]. Proposed by Otto Dunkel, Washington University.

From two fixed points, A and B, of a given conic two chords, AC and BD, are drawn intersecting on the fixed chord, IJ, of the same conic and determining another chord, CD. Determine the envelope of the chord CD and a method for locating points on the envelope without the use of equations.

SOLUTION BY OTTO DUNKEL, Washington University.

This solution includes a generalization of **3138 [1925, 261]**. In that problem the conic is a circle with AB as a diameter, while the variable vertex Q of the triangle QAB moves on a fixed line $l \equiv IJ$. The feet C and D of the altitudes BC and AD must lie on this fixed circle. Here the circle and its diameter are replaced by any given conic and any one of its chords. The line l in this discussion may or may not cut the conic in real points.

Consider two points Q_1 and Q on l and the two sets of lines Q_1C_1A, Q_1D_1B and QCA, QDB which determine, respectively, the chords C_1D_1 and CD of the given conic α. Denote by U the intersection of AQ_1 and BQ; by V the intersection of AQ and BQ_1; by P' the intersection of C_1D_1 and CD. Then from the inscribed hexagon $D_1C_1ACDBD_1$, it follows that U, P', V lie on a straight line cutting AB in E. If l cuts AB in S, then from the complete quadrilateral determined by U, Q_1, V, Q it is seen that S and E are harmonically separated by A and B. Suppose now that Q_1 is fixed, then C_1D_1 is fixed. Let CD cut AB in R. Since Q and R are conjugate points with respect to α, they describe projective ranges on l and AB, respectively. The range Q on l is projective with the range U on AC_1, since UQ passes through B. Also the range U is projective with the range P' on C_1D_1, since UP' passes through E. Hence CD cuts the two fixed lines C_1D_1 and AB in corresponding pairs of points P' and R of two projective ranges; it therefore envelopes a conic β tangent to AB and C_1D_1. When Q is at S, R is at E. Hence E, the limit point of the intersection of two tangents, is the point of tangency of AB.

Now let CD coincide with C_1D_1; then P' becomes the point of tangency P_1 of C_1D_1. Hence P_1 is the intersection of C_1D_1 with Q_1E, and thus we have a simple construction for the point of tangency P of any chord CD with β.

If α and β intersect in any point P, this point must lie on a chord CD and the corresponding line QE; but this is impossible unless P and Q coincide. Hence α and β can intersect only where l cuts α. If l cuts α in two distinct points, then α and β have double contact at these points. This is useful in a discussion of the nature of β.

Having constructed two chords C_1D_1, CD and their points of tangency P_1, P, and the point of tangency E of AB, we may abandon the rest of the figure and construct points of β as follows. Let C_1D_1 and CD cut AB in X and Y, respectively, and draw the line PP_1. Draw any line YX_1 cutting XP_1 in X_1 and PP_1 in Z; draw XZ cutting YP in Y_1; then X_1Y_1 is a tangent to β and the point E_1 in which it is cut by EZ is the point of tangency. This follows easily from Brianchon's theorem.

Let the tangents to α at A and B be TAT_1 and TBT_2, where T is their intersection and T_1 and T_2 lie on l; let T_1B and T_2A meet α in A_1 and B_1. Then AA_1A_2 and BB_1B_2 are tangents to β cutting l in A_2 and B_2; their points of tangency P_a and P_b are their intersections, respectively, with T_1E and T_2E. Then it follows that $AP_aA_1A_2$ and $BP_bB_1B_2$ are each harmonic sets. Also, if T_1B and T_2A meet in M, and AA_1 and BB_1 meet in N, then N, M, T and E lie on a straight line, as shown above, and this line is the polar of S with respect to α. This line contains also a fifth point L, the pole of l. It then follows

that A_1B_1 passes through S and hence P_aP_b also passes through S. Hence S and TE are also pole and polar with respect to β.

Returning now to the figure determined by the variable vertex Q, let BC and AD meet in K, then the polar of R, TQ, passes through K, and also the polar of Q passes through L, K, and R. The figure shows that the pencils $A(D)$, $B(C)$, $L(R)$, $T(Q)$ are projective, and hence the locus of K is a conic γ passing through A, B, L and T. If l cuts α, then γ passes through the points of intersection. It follows also that AA_1 and BB_1 are tangents to γ at A and B; also that SL and ST are tangents at L and T. Hence S and TL are pole and polar for γ.

Problem **3138** was also solved by THEODORE BENNETT, H. W. BAILEY, MICHAEL GOLDBERG, W. J. PATTERSON, AUGUST SÖRENSON, MABEL M. YOUNG, and the PROPOSER (two solutions).

3142 [1925, 315]. Proposed by Harry Langman, New York City.

Cut a rectangle 1×2 into three pieces which will fit into a Maltese cross.

SOLUTION BY ABIGAIL E. JOHNSON, Morristown, N. J.

Let XY be the base of length 2 of rectangle $XYZW$. Using X as the origin and XY and XW as axes of coordinates, divide the base into ten and the altitude into five equal units, and connect the following points in order: (0, 0); (1, 3); (4, 2); (5, 5); (8, 4); (7, 1); (10, 0). The resulting three pieces can be formed into a Maltese cross.

3143 [1925, 315]. Proposed by Edward Condon, University of California.

Prove that

$$\frac{\sin(n-1)\alpha}{\sin n\alpha} = \frac{1}{2\cos\alpha -}\,\frac{1}{2\cos\alpha -}\,\frac{1}{2\cos\alpha -}\cdots,$$

in which the continued fraction terminates when $2\cos\alpha$ has appeared $n-1$ times. Prove also the corresponding formula for the hyperbolic sines and cosines.

SOLUTION BY R. H. SCIOBERETI, Berkeley, California.

The proof of this relation may be based on a classic recursion formula of trigonometry, namely:

$$\sin(n+1)\alpha = 2\cos\alpha\cdot\sin n\alpha - \sin(n-1)\alpha,\quad n=2,3\cdots, \tag{1}$$

from which we derive by division by $\sin n\alpha$,

$$\frac{\sin(n+1)\alpha}{\sin n\alpha} = 2\cos\alpha - \cfrac{1}{\cfrac{\sin n\alpha}{\sin(n-1)\alpha}}.$$

Applying equation (1) to the ratio $\sin n\alpha/\sin(n-1)\alpha$ and repeating this process n times in succession, we shall have converted the quotient $\sin(n-1)\alpha/\sin n\alpha$ into the proposed terminating continued fraction. The given formula can now be proved by induction.

The recursion formula

$$\cos(n+1)\alpha = 2\cos\alpha\cdot\cos n\alpha - \cos(n-1)\alpha \tag{2}$$

shows that the ratio $\cos(n-1)\alpha/\cos n\alpha$ can be converted into a terminating continued fraction of the same form; the last partial quotient is, however, equal to $\cos\alpha$. Hence

$$\frac{\cos(n-1)\alpha}{\cos n\alpha} = \frac{1}{2\cos\alpha -}\,\frac{1}{2\cos\alpha -}\,\frac{1}{2\cos\alpha -}\cdots\frac{1}{-2\cos\alpha}\,\frac{1}{-\cos\alpha},$$

where $\cos\alpha$ appears n times.

As for the hyperbolic functions $\cosh\alpha$ and $\sinh\alpha$, it will be sufficient to observe that the two equations (1) and (2) will be exactly of the same form, since if (1) is multiplied by $i=\sqrt{-1}$ and if α is changed into $(i\alpha)$ in both, then the desired relations follow at once:

$$\sinh(n+1)\alpha = 2\cosh\alpha\sinh n\alpha - \sinh(n-1)\alpha,$$
$$\cosh(n+1)\alpha = 2\cosh\alpha\cosh n\alpha - \cosh(n-1)\alpha.$$

Also solved by MICHAEL GOLDBERG and N. PETROFF.

3145 [1925, 433]. Proposed by W. D. Cairns, Oberlin College.

The center of gravity of any zone of a certain surface of revolution lies midway between the bases of the zone. What is the surface?

SOLUTION BY T. E. STERN, Princeton University.

Consider the surface as generated by the rotation of a curve about the x axis. Take as origin the projection on the x axis of any point on the curve. Consider the curve in the interval Ox. By the given conditions

$$\int_0^x xy\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}dx = \frac{x}{2}\int_0^x y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}dx. \tag{1}$$

Differentiating with respect to x and reducing,

$$\frac{y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}}{\int_0^x y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}dx} - \frac{1}{x} = 0, \tag{2}$$

where the numerators are the x-derivatives of the denominators. Integrating,

$$\int_0^x y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}dx = Kx \tag{3}$$

where k is a constant. Differentiating, and reducing, we get

$$\frac{dy}{dx} = \left[\frac{k^2}{y^2}-1\right]^{1/2}. \tag{4}$$

If y is a constant then $y^2=k^2$. If y is not a constant, then by solving (1), $(x-c)^2+y^2=k^2$, c being a constant. Thus the surface must be either cylindrical or spherical.

Also solved by R. H. SCIOBERETI.

NOTES AND NEWS

Readers are invited to contributed to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

In recognition of the greetings sent by the Mathematical Association to MITTAG-LEFFLER on the occasion of his eightieth birthday anniversary last winter, he has sent to the Association his picture with the inscription: *Propter honorem, quem mihi octoginta annos nato ante diem XVI. Kal. Apr. A.D. MCMXXVI benigne tribuisti, cuiusque memoria animo meo numquam excidere poterit, maximas tibi gratias ago.*

Assistant Professor F. W. OWENS, of Cornell University, has been appointed head of the mathematics department at Pennsylvania State College.

Dr. B. H. CAMP, professor of mathematics at Wesleyan University, was a non-resident lecturer in mathematical statistics in the University of Michigan Summer Session. He gave an elementary and an advanced course in this subject. Dr. Camp recently completed a year of study under Karl Pearson.

Dr. S. D. WICKSELL, of the University of Lund, Sweden, has been appointed special lecturer at the University of Michigan, to be in residence during the entire second semester beginning in February, 1927. Professor Wicksell will give an advanced course in frequency and correlation methods, a course in stellar statistics, and a seminar on research in various lines of application of mathematical statistics.

The retiring address of Professor G. D. BIRKHOFF of Harvard University as President of the American Mathematical Society to be read at the Annual Meeting in conjunction with the sessions of the American Association for the Advancement of Science will have as title: "A mathematical critique of some physical theories."

Professor H. L. RIETZ, of the University of Iowa, has been appointed actuary of the Pension Commission of Chicago for the purpose of investigating the pension systems of the city.

Dr. H. W. TAYLOR of the University of Iowa has been appointed professor of mathematics at Hamline University.

Associate Professor G. SIMON of Adelbert College, Western Reserve University, has been promoted to a full professorship of mathematics.

Mr. A. W. Richeson has been appointed assistant professor of mathematics at the University of Maryland.

Professor H. S. UHLER has resigned as head of the department of physics at Gettysburg College to become associate professor of physics in the Sheffield Scientific School of Yale University.

The following apointments to instructorships are announced:

Rutgers University, Mr. M. G. GALBRAITH, Adelbert College, Western Reserve University, Mr. T. F. COPE, State Teachers College, Oshkosh, Wisconsin, Miss JULIA REGER, Junior College of Sheldon, Iowa, Miss MARY PETERS.

Professor A. A. HEINZ, of Tsing Hua College, Peking, China, died suddenly June 11, at Riverside, California, within a few days after his return to America.

A BRIEF COURSE IN ANALYTIC GEOMETRY
and the Elements of Curve-Fitting

By WALTER B. FORD, *with the coöperation of* RAYMOND W. BARNARD, *University of Chicago*

A textbook in plane and solid analytic geometry, adapted to the freshman course in Analytic Geometry. A chapter on the elements of curve-fitting is of especial value to students of Engineering and Statistics. The "phantom figures" are also very helpful. The latest printing includes seventeen pages of Supplementary Exercises without answers. $2.40

COLLEGE ALGEBRA, Revised Edition

By H. L. RIETZ *and* A. R. CRATHORNE

Special features are the review of high school algebra; the selection and omission of material, the full statement of the assumption on which proofs are based, and the application of algebraic methods to physical problems. The applications of algebra in advanced mathematics are duly indicated. Several hundred new exercises and problems have been introduced. $1.76

MATHEMATICS OF FINANCE

By H. L. RIETZ, A. R. CRATHORNE, *and* J. C. RIETZ

Treats in particular the relation of interest to the amortization of debts, to the creation of sinking funds, to the treatment of depreciation, to the valuation of bonds, to the accumulation of funds in building and loan associations, and to the elements of life insurance. Adopted in many leading universities, it is also adapted to the schools and colleges of commerce and business administration. $3.00

HENRY HOLT AND COMPANY
One Park Avenue, New York

Announcing a New Edition of Alexander Smith's

INTRODUCTION TO INORGANIC CHEMISTRY

Smith's Inorganic Chemistry

Revised and Rewritten By

JAMES KENDALL

Professor of Chemistry at New York University

Alexander Smith's "Introduction to Inorganic Chemistry" has been a standard textbook for many years. In preparing the new edition of this widely used book, Dr. Kendall retained all the characteristic Smith features, clarifying explanations and making such additions and changes only as are required by recent discoveries. The new text is up to date in every respect.

The new edition will be published early in the fall. Professors desiring to inspect copies of the book with a view to adoption are invited to write for examination copies.

353 Fourth Ave. New York **THE CENTURY CO.** 2126 Prairie Ave. Chicago

CONTENTS

The Mathematical Association. By W. D. CAIRNS 343
A New Publication of the Association 344
Affiliation of the Association of Teachers of Mathematics in New England with the Mathematical Association of America. By W. D. CAIRNS 346
Annual Meeting of the Rocky Mountain Section. By PHILIP FITCH 346
Eleventh Annual meeting of the Ohio Section. By RUFUS CRANE 350
Third Annual Meeting of the Michigan Section. By NORMAN ANNING 352
On the Asymptotic Expressions for Jacobi and Legendre Polynomials Derived from Finite-difference Equations. By J. A. SHOHAT 354
Some Applications of Mathematics to Architecture: Gothic Tracery Curves. By E. C. PHILLIPS 361
On the De Longchamps Circle of the Triangle. By NATHAN ALTSHILLER-COURT 368
QUESTIONS AND DISCUSSIONS: Discussions—"Mathematical induction" by by G. E. RAYNOR; "A new curve connected with two classic problems" by G. M. JUREDINI; "Note on a linear Diophantine equation" by ELIZABETH B. COWLEY 376
RECENT PUBLICATIONS: Reviews by H. L. RIETZ, DAVID EUGENE SMITH. Articles in current periodicals 382
PROBLEMS AND SOLUTIONS: Problems for solution—**3206–3211**; Solutions—**2827, 3118, 3138, 3152, 3142, 3143, 3145** 385
NOTES AND NEWS 389

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Tenth Summer Meeting of the Association, Columbus, Ohio, September 7-8, 1926.

Eleventh Annual Meeting, Philadelphia, Pa., December, 30-31, 1926.

The following are dates of Section Meetings of the Association in 1926:

ILLINOIS, Decatur, Ill., May 7-8.

INDIANA, Purdue University, May, 7–8.

IOWA, Cedar Rapids, April.

KANSAS, Merged in National Meeting.

KENTUCKY, Berea College, May 1.

LOUISIANA-MISSISSIPPI, New Orleans, La., March 12-13.

MARYLAND - DISTRICT OF COLUMBIA - VIRGINIA, Annapolis, Md., December 4.

MICHIGAN, Ann Arbor, Mich., April 1.

MINNESOTA, Northfield, Minn., May 22.

MISSOURI, Kansas City, Mo., November.

NEBRASKA, Bethany, Neb., May.

OHIO, Columbus, Ohio, April 2.

ROCKY MOUNTAIN, Colorado College, April, 1927.

SOUTHEASTERN, Atlanta, Ga., March 19-20.

SOUTHERN CALIFORNIA, Los Angeles, Calif., November 6.

TEXAS, November.

Secretaries of Sections will please report changes or corrections promptly to the Editor.

343
344

346
346
350
352

354

361

368

376

382

385
389

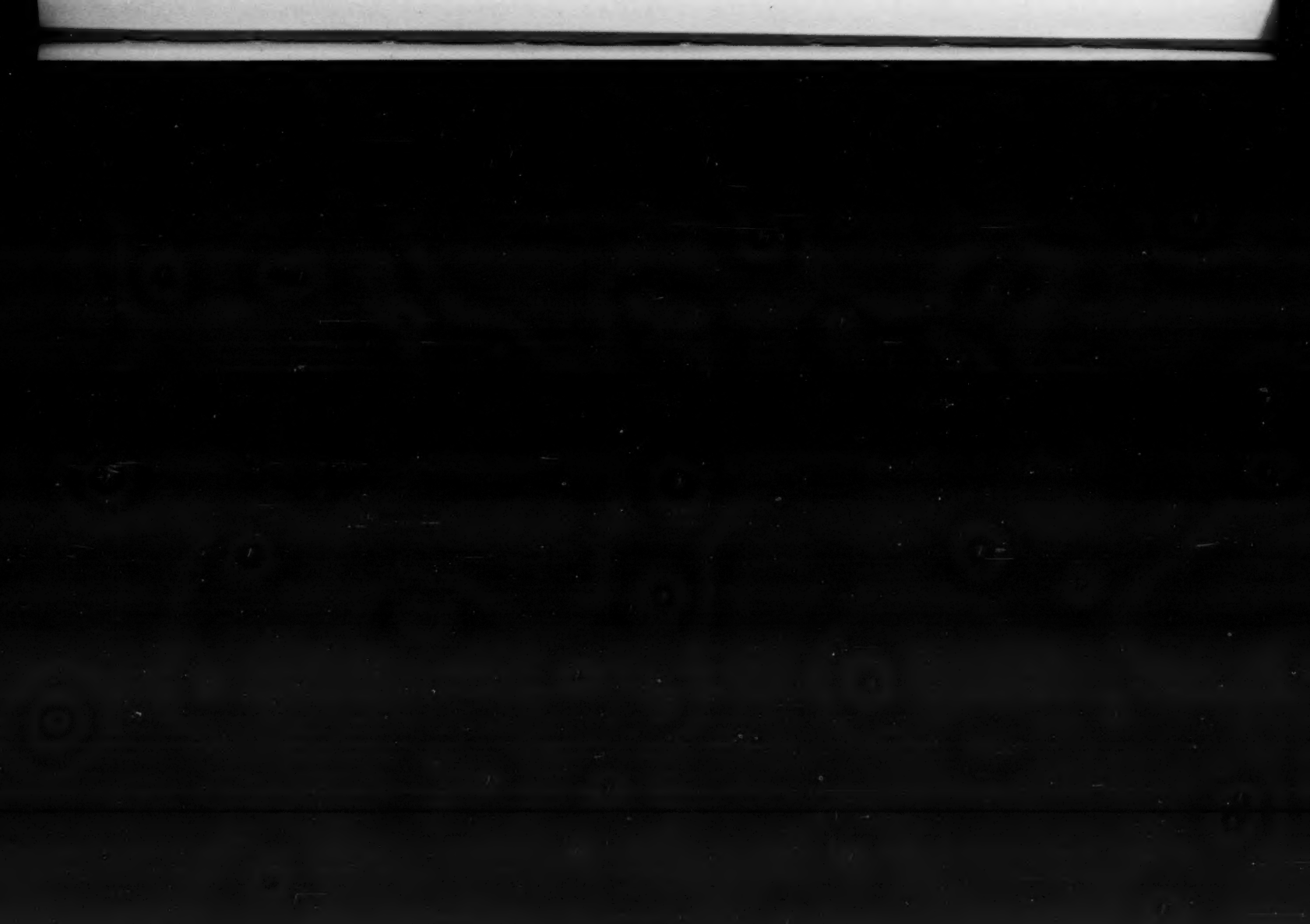
IEF,

Y.
RER

pril,

-20.
lif.,

r.



THE MAY MEETING OF THE ILLINOIS SECTION

The seventh annual meeting of the Illinois Section of the Mathematical Association of America was held at James Millikin University, Decatur, on May 7–8, 1926. Chairman E. B. Lytle presided. A short address of welcome was given by President E. Penney of James Millikin University.

The attendance was thirty-nine including the following twenty-eight members of the Association:

Beulah Armstrong, R. D. Carmichael, C. E. Comstock, D. R. Curtiss, Arnold Emch, A. E. Gault, Mary G. Haseman, Mabel M. Herren, Mildred Hunt, E. C. Kiefer, E. B. Lytle, W. D. MacMillan, Martha P. McGavock, R. M. Mathews, Bessie I. Miller, C. N. Mills, E. J. Moulton, Rev. Paul Muehlman, Mary W. Newson, H. P. Pettit, Theresa M. Renner, G. T. Sellew, H. A. Simmons, H. E. Slaught, C. J. Stowell, Mildred E. Taylor, M. E. Wescott, F. E. Wood.

The following officers were elected for the next year: Chairman, E. C. KIEFER, James Millikin University; Vice-chairman, H. P. PETTIT, Illinois Wesleyan University; Secretary-Treasurer, BESSIE I. MILLER, Rockford College. The executive committee was advised to accept the invitation of Professor Pettit to have the next meeting of the section at Illinois Wesleyan University. The secretary was instructed to extend to Professor Kiefer and to James Millikin University a unanimous vote of thanks for the hearty welcome given the Section and for the very pleasant arrangements which had been made for the holding of the meeting.

The following program was presented:

1. "Diophantine problems in weighing" by Professor H. A. SIMMONS, Northwestern University.
2. "Mathematics of art and literature" (Illustrated) by Professor ARNOLD EMCH, University of Illinois.
3. "College geometry" by Professor R. M. MATHEWS, University of Illinois.
4. "Discussion: Undergraduate courses in geometry" led by Professor BESSIE I. MILLER, Rockford College.
5. "The Einstein theory" by Professor R. D. CARMICHAEL, University of Illinois.
6. "The second Carus monograph" by Professor D. R. CURTISS, Northwestern University.
7. "What constitutes a good teacher of mathematics?" by Professor MARY W. NEWSON, Eureka College.